Interpolation

Introduction

1. For analyzing functions $f(x)$, say finding minima, we use a fundamental assumption that we can obtain $f(x)$ when we want it, regardless of x. There are many contexts in which this assumption is *unrealistics.*
2. We need a model for interpolating $f(x)$ to all of $\mathbb{R}^n$ given a collection of samples $f(x_i)$
3. We seek for the interpolated function (also denoted as $f(x)$) to be smooth and serve as a reasonable prediction of function values.
4. We will design methods for interpolating functions of single variable, using the set of polynomials.

Interpolation

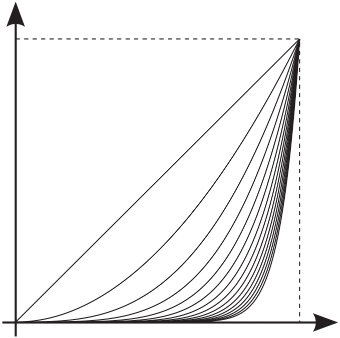
Polynomial representation in a basis:

$$f(x) = a_1\phi_1(x) + a_2\phi_2(x) + \cdots + a_k\phi_k(x)$$

where $\{\phi_1(x), \phi_2(x), \ldots, \phi_k(x)\}$ is a basis:

1. Monomial basis:

$$\phi_i(x) = x^{i-1}$$



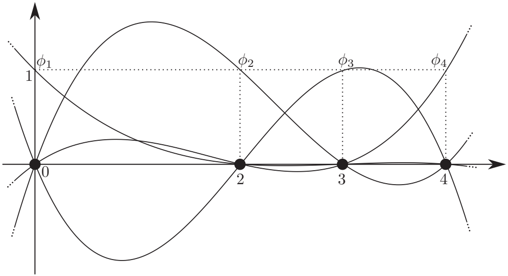
Interpolation

2. Lagrange basis

$$\phi_i(x) = \frac{\prod_{j \neq i}(x - x_j)}{\prod_{j \neq i}(x_i - x_j)}$$

where $\{x_1, x_2, \ldots, x_k\}$ are prescribed distinct points. Note that

$$\phi_i(x_\ell) = \begin{cases} 1 & \text{when } \ell = i \\ 0 & \text{otherwise} \end{cases}$$



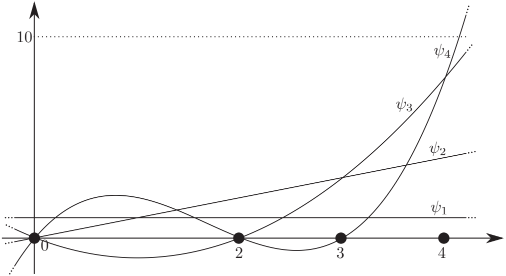
Interpolation

3. Newton basis

$$\phi_i(x) = \prod_{j=1}^{i-1}(x - x_j) \quad \text{with} \quad \phi_1(x) \equiv 1,$$

where $\{x_1, x_2, \ldots, x_k\}$ are prescribed distinct points.
Note that

$$\phi_i(x_\ell) = 0 \quad \text{for all } \ell < i.$$



Interpolation

Polynomial interpolation:

Given a set of k points (x_i, y_i), with the assumption $x_i \neq x_j$.
Find a polynomial $f(x)$ of degree $k-1$ such that $f(x_i) = y_i$.

Interpolation

1. **Interpolating polynomial in monomial basis**

$$f(x) = a_1 + a_2 x + a_3 x^2 + \cdots + a_k x^{k-1}$$

where $a_1, a_2, \ldots, a_k$ are determined by the Vandermonde linear system:

$$\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{k-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{k-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_k & x_k^2 & \cdots & x_k^{k-1} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_k \end{bmatrix}$$

2. **Interpolating polynomial in Lagrange basis**

$$f(x) = y_1 \phi_1(x) + y_2 \phi_2(x) + \cdots + y_k \phi_k(x)$$

Interpolation

3. **Interpolating polynomial in Newton basis**

$$f(x) = a_1\phi_1(x) + a_2\phi_2(x) + \cdots + a_k\phi_k(x)$$

where $a_1, a_2, \ldots, a_k$ are determined by the following triangular systems:

$$\begin{bmatrix} 1 & & & \\ 1 & \phi_2(x_2) & & \\ \vdots & \vdots & \ddots & \\ 1 & \phi_2(x_k) & \cdots & \phi_k(x_k) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_k \end{bmatrix}$$

Interpolation

Remarks

1. The Verdermonde system could be poor conditioned and unstable.
2. Computing $f(x)$ in Lagrange basis takes $O(k^2)$ time, constrastingly, computing $f(x)$ in monomial basis takes only $O(k)$ by Horner's rule.
3. $f(x)$ in Newton basis attempts to compromise between the numerical quality of the monomial basis and the efficiency of the Lagrange basis.

Examples

- `interpeg1.m`
- `interpeg2.m`
- `interpeg3.m`

Piecewise interpolation

1. So far, we have constructed interpolation bases defined on all of $\mathbb{R}$.
2. When the number k of data points becomes large, many degeneracies apparent. Mostly noticble, the polynomial interpolation is *nonlocal*, changing any single value y_i can change the behavior of $f(x)$ for all x, even those that are far away from x_i. This property is undersiable from most applications.
3. A solution to avoid such drawback is to design a set of base functions $\phi_i(x)$ of the property of *compact support*:

   > *A function $g(x)$ has* compact support *if there exists a constant $c \in \mathbb{R}$ such that $g(x) = 0$ for any x with $\|x\|_2 > c$.*

4. Piecewise formulas provide one technique for constructing interpolatory bases with compact support.

Piecewise interpolation

Piecewise constant interpolation:

1. Order the data points such that $x_1 < x_2 < \cdots < x_k$
2. For $i = 1, 2, \ldots, k$, define the basis

$$\phi_i(x) = \begin{cases} 1 & \text{when } \frac{x_{i-1}+x_i}{2} \leq x < \frac{x_i+x_{i+1}}{2} \\ 0 & \text{otherwise} \end{cases}$$

3. Piecewise constant interpolation

$$f(x) = \sum_{i=1}^{k} y_i \phi_i(x)$$

4. discontinuous!

Piecewise interpolation

Piecewise **linear** **interpolation**:

1. Order the data points such that $x_1 < x_2 < \cdots < x_k$
2. Define the basis ("hat functions")

$$\phi_i(x) = \begin{cases} \frac{x - x_{i-1}}{x_i - x_{i-1}} & \text{when } x_{i-1} < x \leq x_i \\ \frac{x_{i+1} - x}{x_{i+1} - x_i} & \text{when } x_i < x \leq x_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

   for $i = 2, \ldots, k-1$ with the boundary "half-hat" basis $\phi_1(x)$ and $\phi_k(x)$.

3. Piecewise linear interpolation

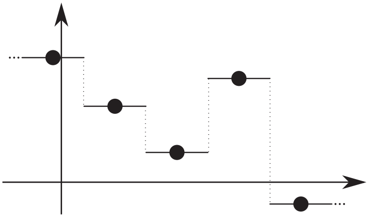
$$f(x) = \sum_{i=1}^{k} y_i \phi_i(x)$$

4. Continuous, but non-smooth.
5. Smooth piecewise high-degree polynomial interpolation – "splines"

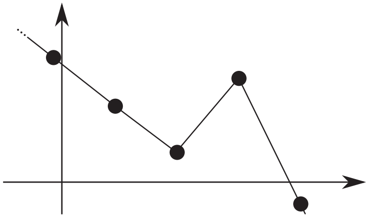
Piecewise interpolation

Piecewise constant

Piecewise linear



Piecewise constant



Piecewise linear

Theory of interpolation

1. Linear algebra of functions
2. Error bound of piecewise interpolations

Theory of interpolation

Linear algebra of functions

1. There are other bases (beyond monomials, Lagranges and Newtons) for the set of functions f.
2. Inner product of functions f and g:

$$\langle f, g \rangle_w = \int_a^b w(x) f(x) g(x) dx$$

and

$$\|f\| = \sqrt{\langle f, f \rangle_w}$$

where $w(x)$ is a given positive (weighting) function.

Theory of interpolation

3. **Lagendre polynomials**

   Let $a = -1$, $b = 1$ *and* $w(x) = 1$, *applying Gram-Schmidt process to the monomial basis* $\{1, x, x^2, x^3, \ldots\}$, *we generate the Lagendre basis of polynomials:*

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= \frac{1}{2}(3x^2 - 1) \\ P_3(x) &= \frac{1}{2}(5x^3 - 3x), \quad \ldots \end{aligned}$$

   where $\{P_i(x)\}$ *are orthogonal.*

Theory of interpolation

4. An application of Lagendre polynomials:
   Least squares function approximation (not interpolation)

$$\min_{a_i} \|f - \sum_{i=1}^{n} a_i P_i(x)\| = \|f - \sum_{i=1}^{n} a_i^* P_i(x)\|$$

   where

$$a_i^* = \frac{\langle f, P_i \rangle}{\langle P_i, P_i \rangle}.$$

   Note that we need intergration here, numerical integration to be covered later.

Theory of interpolation

5. **Chebyshev polynomials**

   Let $a = -1$, $b = 1$ *and* $w(x) = \frac{1}{\sqrt{1-x^2}}$, *applying Gram-Schmidt process to the monomial basis* $\{1, x, x^2, x^3, \dots\}$, *we generate the Chebyshev basis of polynomials:*

$$T_0(x) = 1$$
$$T_1(x) = x$$
$$T_2(x) = 2x^2 - 1$$
$$T_3(x) = 4x^3 - 3x, \quad \dots$$

   where $\{T_i(x)\}$ *are orthogonal.*

Theory of interpolation

6. Surprising properties of Chebyshev polynomials
   (a) Three-term recurrence

   $$T_{k+1} = 2xT_k(x) - T_{k-1}(x)$$

   with $T_0(x) = 1$ and $T_1(x) = x$.
   (b) $T_k(x) = \cos(k \arccos(x))$
   - ...
7. Chebyshev polynomials play important role in modern numerical algorithms for solving very large scale linear systems and eigenvalue and singular value problems!

Theory of interpolation

Error bound of piecewise interpolations

1. Consider the approximation of a function $f(x)$ with a polynomial of degree n on an interval $[a, b]$. Define $\Delta = b - a$
2. Piecewise constant interpolation
   If we approximate $f(x)$ with a constant $c = f(\frac{a+b}{2})$, as in piecewise constant interpolation, and assume that $|f'(x)| \leq M$ for all $x \in [a, b]$, then
   $$\max_{x\in[a,b]} |f(x) - c| \leq M\Delta x = O(\Delta x)$$

Theory of interpolation

3. Piecewise linear interpolation

   Approximate $f(x)$ with

$$\widetilde{f}(x) = f(a)\frac{b-x}{b-a} + f(b)\frac{x-a}{b-a}.$$

   By the Taylor series

$$f(a) = f(x) + (a-x)f'(x) + \cdots$$
$$f(b) = f(x) + (b-x)f'(x) + \cdots$$

   we have

$$\widetilde{f}(x) = f(x) + \frac{1}{2}(x-a)(x-b)f''(x) + O((\Delta x)^3).$$

   Therefore, the error $= O(\Delta x^2)$ assuming $f''(x)$ is bounded. Note that $|x-a|\,|x-b| \leq \frac{1}{2}(\Delta x)^2$.