1. **LU decomposition (Gaussian elimination in matrix form).** If A is a square nonsingular matrix, then there exist a permutation matrix P, a unit lower triangular matrix L, and a upper triangular matrix U such that

$$PA = LU.$$

Special cases:

(a) **Cholesky decomposition**. A matrix A is symmetric positive definite *if and only if* there exists a unique nonsingular upper triangular matrix R, with positive diagonal entries, such that

$$A = R^T R.$$

(b) **LDL$^{\mathrm{T}}$ factorization.** If $A^T = A$ is nonsingular, then there exists a permutation P, a unit lower triangular matrix L, and a block diagonal matrix D with 1-by-1 and 2-by-2 blocks such that

$$PAP^T = LDL^T.$$

Applications:

- Solve $Ax = b$.
- Compute $\det(A)$.
- Compute A^{-1}, if really necessary.

2. **QR decomposition.** Let A be m-by-n with $m \geq n$. Suppose that A has full column rank. Then there exist a unique m-by-n orthogonal matrix Q (i.e. $Q^T Q = I$) and a unique n-by-n upper triangular matrix R with positive diagonal $r_{ii} > 0$ such that

$$A = QR.$$

Applications:

- Find an orthonormal basis of the subspace spanned by the columns of A (the Gram-Schmidt orthogonalization process)
- Solve the linear least squares problem $\min_x \|Ax - b\|_2$.

3. **Schur decomposition, eigenvalue decomposition and spectral decomposition.** Let A be of order n. Then

(a) there is an $n \times n$ unitary matrix U (i.e. $U^H U = I$) such that

$$A = UTU^H,$$

where T is upper triangular. This is called a **Schur decomposition**.

(b) The *eigenvalue decomposition*, if exists, is given by

$$A = X \Lambda X^{-1},$$

where Λ is a diagonal matrix.

(c) When A is Hermitian, $A^H = A$, we have the *spectral decomposition*

$$A = Q\lambda Q^H,$$

where Λ is real and diagonal.

Applications:

- The eigenvalues of A are the diagonal elements of T. By appropriate choice of U, the eigenvalues of A, which are the diagonal elements of T, may be made to appear in any order.
- Compute matrix functions $f(A) = Uf(T)U^H$.

4. **Singular Value Decomposition (SVD).** Let A be an m-by-n matrix with $m \geq n$. Then we can write

$$A = U\Sigma V^T,$$

where U is m-by-m orthogonal matrix (i.e. $U^T U = I_m$) and V is n-by-n orthogonal matrix (i.e. $V^T V = I_n$), and $\Sigma = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n)$, where $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$.

If $m < n$, the SVD can be defined by considering A^T.

The columns $u_1, u_2, \ldots, u_n$ of U are called *left singular vectors* of A. The columns $v_1, v_2, \ldots, v_n$ of V are called *right singular vectors*. The $\sigma_1, \sigma_2, \ldots, \sigma_n$ are called *singular values*.

Applications:

- Suppose that A is m-by-n with $m \geq n$ and has full rank, with $A = U\Sigma V^T$ being A's SVD. Then the pseudo-inverse can also be written as

$$A^\dagger \equiv (A^T A)^{-1} A^T = V\Sigma^{-1} U^T.$$

If $m < n$, then $A^\dagger = A^T(AA^T)^{-1}$.

- Suppose that

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > \sigma_{r+1} = \cdots = \sigma_n = 0,$$

Then the rank of A is r. The range space of A is $\text{span}(u_1, u_2, \cdots, u_r)$. and the null space of A is $\text{span}(v_{r+1}, v_{r+2}, \ldots, v_n)$.

- $\|A\|_2 = \sigma_1 (\equiv \sigma_{\max})$
- Let A be $m \times n$ with $m \geq n$. Then
(a) eigenvalues of $A^T A$ are σ_i^2, $i = 1, 2, \ldots, n$. The corresponding eigenvectors are the right singular vectors v_i, $i = 1, 2, \ldots, n$.
(b) eigenvalues of AA^T are σ_i^2, $i = 1, 2, \ldots, n$ and $m - n$ zeros. The left singular vectors u_i, $i = 1, 2, \ldots, n$ are corresponding eigenvectors for the eigenvalues σ_i^2. One can take any $m - n$ other orthogonal vectors that are orthogonal to $u_1, u_2, \ldots, u_n$ as the eigenvectors for the eigenvalues 0.
- Principal components. The SVD of A can be rewritten as

$$A = E_1 + E_2 + \cdots + E_p$$

where $p = \min(m, n)$, and E_k is a rank-one matrix of the form

$$E_k = \sigma_k u_k v_k^T,$$

E_k are referred to as component matrices, and are orthogonal to each other in the sense that

$$E_j E_k^T = 0, \quad j \neq k.$$

Since $\|E_k\|_2 = \sigma_k$, the contribution each E_k makes to reproduce A is determined by the size of the singular value σ_k.

- Optimal rank-k approximation:

$$\min_{\substack{B\,:\,m\times n\\ \operatorname{rank}(B)=k}} \|A-B\|_2 = \|A-A_k\|_2 = \sigma_{k+1},$$

where

$$A_k = U\Sigma_k V^T, = E_1 + E_2 + \cdots + E_k,$$

and $\Sigma_k = \operatorname{diag}(\sigma_1, \sigma_2, \ldots, \sigma_k, 0, \ldots, 0)$ [1]

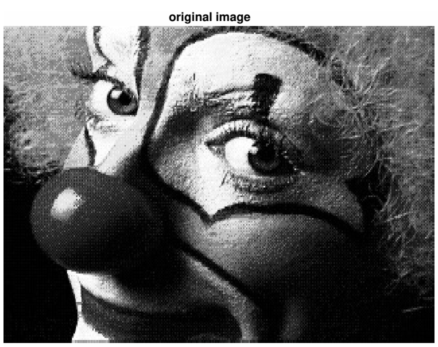
- Data compression. Note that the optimal rank-k approximation A_k can be written in a compact form as

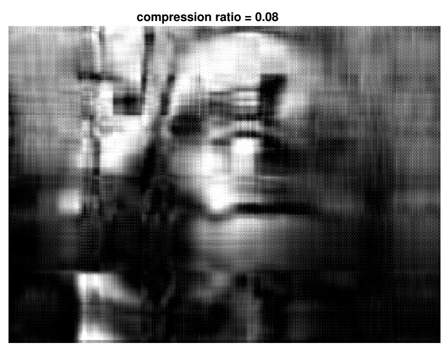
$$A_k = U_k \widehat{\Sigma}_k V_k^T,$$

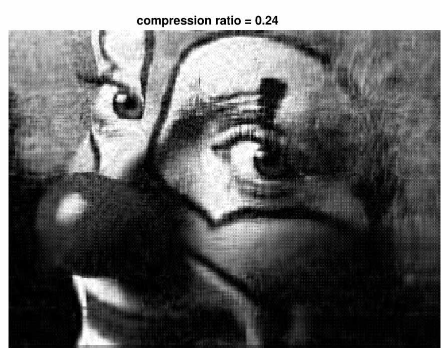
where U_k and V_k are the first k columns of U and V, respectively, $\widehat{\Sigma}_k = \operatorname{diag}(\sigma_1, \sigma_2, \ldots, \sigma_k)$. Therefore, A_k is represented by $mk + k + nk = (m+n+1)k$ elements, in contrast, A is represented by mn elements.

$$\text{compression ratio} = \frac{(m+n+1)k}{mn}$$

The following plots show the original image, and three compressed ones with different compression ratios:







[1] In [Golub, Hoffman and Stewart, LAA, vol.88/89, pp.317-327, 1987], it is shown how to obtain a best approximation of lower rank in which a specified set of columns of the matrix A remains fixed.