ECS231 Handout **Vector and Matrix Norms**

Norms are an indispensable tool to provide vectors and matrices with measures of size, length and distance.

I. Vector norms

1. A *vector norm* on $\mathbb{C}^n$ is a mapping that maps each $x \in \mathbb{C}^n$ to a real number $\|x\|$, satisfying

   (a) $\|x\| > 0$ for $x \neq 0$, and $\|0\| = 0$ (positive definite property)
   (b) $\|\alpha x\| = |\alpha|\,\|x\|$ for $\alpha \in \mathbb{C}$ (absolute homogeneity)
   (c) $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality)

2. Vector p-norm:

$$\|x\|_p \stackrel{\text{def}}{=} \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p},$$

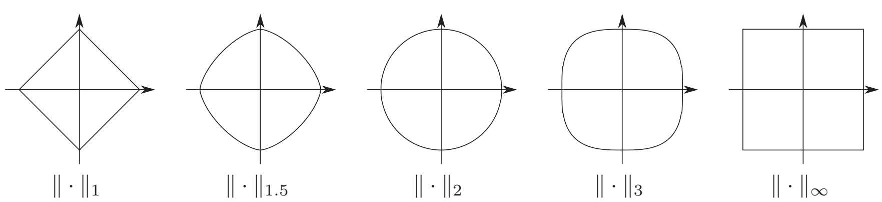
   where $1 \leq p \leq \infty$.

3. Commonly used vector norms:

$$\begin{aligned} \|x\|_1 &= \sum_{i=1}^{n} |x_i|, \quad \text{“Manhattan” or “taxi cab” norm} \\ \|x\|_2 &= \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2} = \sqrt{x^H x}, \quad \text{Euclidean length} \\ \|x\|_\infty &= \max_{1 \leq i \leq n} |x_i|. \end{aligned}$$

4. The geometry of the closed unit “ball”:

$$\{x \in \mathbb{C}^2 : \|x\|_p \leq 1\} \; for \; p = 1, 2, \infty.$$



5. Norm equivalence: Let $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ be any two vector norms, then there are constants $c_1, c_2 > 0$ such that

$$c_1 \|\cdot\|_\alpha \leq \|\cdot\|_\beta \leq c_2 \|\cdot\|_\alpha$$

   For examples, it can be easily shown that

$$\|x\|_\infty \leq \|x\|_2 \leq \sqrt{n}\|x\|_\infty$$

$$\|x\|_2 \leq \|x\|_1 \leq \sqrt{n}\|x\|_2$$

$$\|x\|_\infty \leq \|x\|_1 \leq n\|x\|_\infty$$

6. Cauchy-Schwarz inequality:

$$|x^H y| \leq \|x\|_2 \|y\|_2$$

   with equality if and only if x and y are linearly dependent. In general, Hölder inequality:

$$|x^H y| \leq \|x\|_p \|y\|_q \quad \text{for } 1 \leq p, q < \infty \text{ and } \frac{1}{p} + \frac{1}{q} = 1.$$

II. Matrix norms

1. A *matrix norm* on $\mathbb{C}^{m\times n}$ is a mapping that maps each $A \in \mathbb{C}^{m\times n}$ to a real number $\|A\|$, satisfying

   (a) $\|A\| > 0$ for $A \neq 0$, and $\|0\| = 0$ (positive definite property)

   (b) $\|\alpha A\| = |\alpha|\,\|A\|$ for $\alpha \in \mathbb{C}$ (absolute homogeneity)

   (c) $\|A + B\| \leq \|A\| + \|B\|$ (triangle inequality)

2. Example: for $A = (a_{ij}) \in \mathbb{C}^{m\times n}$, the Frobenius norm $\|A\|_{\mathrm{F}}$ is defined by

$$\|A\|_{\mathrm{F}} \stackrel{\text{def}}{=} \left(\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2\right)^{1/2} = \sqrt{\mathrm{tr}(A^H A)}.$$

3. The *induced matrix norm* $\|\cdot\|$:

   A vector norm $\|\cdot\|$ induces a matrix norm, denoted by

$$\|A\| \stackrel{\text{def}}{=} \max_{x\neq 0}\frac{\|Ax\|}{\|x\|} = \max_{\|x\|=1}\|Ax\|$$

   Exercise. Verify that $\|A\|$ is indeed a norm on $\mathbb{C}^{m\times n}$.

4. A useful property for the induced matrix norm:

$$\|Ax\| \leq \|A\|\,\|x\|.$$

   Therefore, $\|A\|$ is the maximal factor by which A can "strech" a vector.

5. The vector p-norms induce the matrix p-norms for the popular $p = 1, 2, \infty$:

$$\|A\|_1 = \max_{x\neq 0}\frac{\|Ax\|_1}{\|x\|_1} = \max_{1\leq j\leq n}\left\{\sum_{i=1}^{m}|a_{ij}|\right\} = \text{max absolute column sum},$$

$$\|A\|_2 = \max_{x\neq 0}\frac{\|Ax\|_2}{\|x\|_2} = \sqrt{\text{the largest eigenvalue of } A^*A} = \text{the largest singular value of } A,$$

$$\|A\|_\infty = \max_{x\neq 0}\frac{\|Ax\|_\infty}{\|x\|_\infty} = \max_{1\leq i\leq m}\left\{\sum_{j=1}^{n}|a_{ij}|\right\} = \text{max absolute row sum}.$$

6. Some useful properties:

   - $\|A\|_2^2 \leq \|A\|_1\|A\|_\infty$.
   - Norm equivalence

III. An application

1. Sensitivity analysis of linear system of equations $Ax = b$.