ECS231 Problem Set #1 **due 4pm, April 19**

1. Consider the function $f(x) = \log x$. The condition number is $\kappa_f(x) = |1/\log x|$, which is large for $x \approx 1$. Numerically demonstrate that a small relative change in x can be produce a much larger relative change in $\log x$ for $x \approx 1$. Use the rule of thumb:

$$\text{(relative forward error)} \lesssim \text{(condition number)} \times \text{(relative backward error)}.$$

to explain your numerical results.

2. In this problem, we explore the conditioning of root-finding. Suppose $f(x)$ and $p(x)$ are smooth functions of $x \in \mathbb{R}$, and x_* is a root of $f(x)$, i.e., $f(x_*) = 0$.

(a) Due to error in evaluating $f(x)$, we might compute roots of a perturbation $f(x) + \varepsilon p(x)$. If $f'(x_*) \neq 0$, for small ε we can write a function $x(\varepsilon)$ such that $f(x(\varepsilon)) + \varepsilon p(x(\varepsilon)) = 0$ with $x(0) = x_*$. Assuming such a function $x(\varepsilon)$ exists and is differentiable, show that

$$\left.\frac{dx}{d\varepsilon}\right|_{\varepsilon=0} = -\frac{p(x_*)}{f'(x_*)}$$

(b) Consider Wilkinson's polynomial

$$f(x) = (x-1)(x-2)\cdots(x-20).$$

We could have expanded $f(x)$ in the monomial basis as $f(x) = a_0 + a_1 x + \cdots + a_{20}x^{20}$. If we express the coefficient a_{19} in accurately, we could use the model from (a) with $p(x) = x^{19}$ to predict how much root-finding will suffer. Show that

$$\left.\frac{dx}{d\varepsilon}\right|_{\varepsilon=0, x_*=j} = -\prod_{k\neq j} \frac{j}{j-k}$$

(c) Compare $\frac{dx}{d\varepsilon}$ from (b) for $x_* = 1$ and $x_* = 20$. Which root is more stable to the perturbation?

3. The power series for $\sin x$ is

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Here is a Matlab function that uses this series to compute $\sin x$.

```
function s = powersin(x)
% POWERSIN(x) tries to comptue sin(x) from a power series
s = 0;
t = x;
n = 1;
while s + t ~= s;
   s = s + t;
   t = -x.^2/((n+1)*(n+2)).*t;
   n = n + 2;
end
```

(a) What cases the while loop to terminate?

(b) Answer each of the following questions for $x = \pi/2, 11\pi/2, 21\pi/2$ and $31\pi/2$.

- How accurate is the computed results?
- How many terms are required?
- What is the largest term (i.e, the last `t`) in the series?

(c) What do you conclude about the use of floating point arithmetic and power series to evaluate functions?

4. The roots of the quadratic function $ax^2 + bx + c$ are given by

$$x_* = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

(a) Prove the alternative formula

$$x_* = \frac{-2c}{b \pm \sqrt{b^2 - 4ac}}.$$

(b) Propose a numerical stable algorithm for finding the roots.

(c) Compare/comment your computed results for the following set of data:

1) $a = 1$, $b = -56$, $c = 1$

2) $a = 1$, $b = -10^8$, $c = 1$

5. Consider the function

$$f(x) = \frac{e^x - 1}{x},$$

which arises in various applications. By L'Hopital's rule, we know that

$$\lim_{x \to 0} f(x) = 1.$$

(a) Compute the values of $f(x)$ for $x = 10^{-n}$, $n = 1, 2, \ldots, 16$. Do your results agree with theoretical expectations? Explain why.

(b) Perform the computation in part (a) again, this time using the mathematically equivalent formulation

$$f(x) = \frac{e^x - 1}{\log(e^x)},$$

(evaluated as indicated with no simplification). If this works any better, can you explain why?

6. (Mini-project) Program the five algorithms discussed in the class for the matrix-matrix multiply $C = C + AB$, where A, B and C are $n \times n$ matrix. Time them for random matrices for a set of dimensions. Verify that they yield the same solution but takes different amount of time (and different rates of flops).