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Recursion and The Towers of Hanoi

The Towers of Hanoi is a classic puzzle that has fascinated laymen and mathematicians alike for over a hundred years. The riddle was first developed in 1883 by French mathematician Édouard Lucas. The game seems deceptively simple: in the original version, there are 3 pegs. The first peg has a certain number of discs stacked upon it in order of ascending size. The object of the game is to move all of the discs to the third peg. However, the catch is that no disc can be placed on top of a disc that is smaller than itself.

In hopes of making his idea more appealing to the public, Lucas also fabricated a legend that tells of an ancient monastery in which there was a room with a large-scale Towers of Hanoi and 64 golden discs. Brahmin priests at the monastery have dedicated their lives to solving the puzzle. The belief is that once the priests have moved all 64 discs to the final tower, the world will end. Obviously, the biggest question that arises from this legend is, "How long will it take the priests to finish and when will the world end?"

To answer this question, we need to find out how many moves it will take for the priests to solve their 64-disc Towers of Hanoi. This is done by writing a recurrence – $T(n)$ for the number of moves `MoveTower` takes for an n-disk tower. Establishing a base case is simple: if there is only one disk, there only needs to be one move. All other solutions for n number of discs are based off this solution. For instance, take the case of 2 discs, where the starting peg is peg A, the intermediate peg is peg B, and the final peg is peg C.

1. Use the one disc (n - 1) solution to move the top disc to peg B.
2. Move the bottom disc to the peg C.

3. Use the one disc solution to move the top disc to the peg C.

For n discs, one needs to use the n - 1 solution until the preceding algorithm is used for the base case where $n = 1$. The general solution is as follows:

1. Move $n−1$ discs from A to B. This leaves disc n alone on peg A.
2. Move disc n from peg A to peg C.
3. Move n - 1 discs from peg B to peg C.

So, if $T(n)$ represents the number of steps needed for n discs, $T(n) = T(n - 1) + T(1) + T(n - 1) = 2T(n - 1) + 1$. This represents the recursive solution for the Towers of Hanoi, and can also be written as $2^n + 1$.

The above solution is the only one that results in the minimal number of moves required to solve any Towers of Hanoi with n discs. Hence, we can now determine how many moves the priests will have to complete in order to finish their version with 64 discs. If $n = 64$, then $T(n) = 2^{64} + 1$, or approximately 1.8×10^{18} steps. If we assume that the priests can move one disc per second indefinitely, then it will take them about 585 billion years to finish the Towers of Hanoi, or roughly 45 times the life span of the sun.

Today, the Towers of Hanoi is widely used as part of aptitude or IQ tests. It is also commonly utilized in psychological research on problem solving as well as studies on frontal lobe deficits. Interestingly, researchers have found that a certain species of ant in Argentina have demonstrated the ability to solve the Towers of Hanoi. The researchers translated the traditional Towers layout into a maze connecting the ant nest to a food source, and the ants were able to successfully map out the minimal pathway to the food via non-linear dynamics and pheromone signaling. Finally, the Towers of Hanoi is a very popular teaching method to introduce the idea of recursive algorithms to beginning computer science students.