Constructive Dimension and Weak Truth-Table Degrees

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Abstract. This paper examines the constructive Hausdorff and packing dimensions of weak truth-table degrees. The main result is that every infinite sequence S with constructive Hausdorff dimension $\dim_{\mathrm{H}}(S)$ and constructive packing dimension $\dim_{\mathrm{P}}(S)$ is weak truth-table equivalent to a sequence R with $\dim_{\mathrm{H}}(R) \geq \dim_{\mathrm{H}}(S)/\dim_{\mathrm{P}}(S) - \epsilon$, for arbitrary $\epsilon > 0$. Furthermore, if $\dim_{\mathrm{P}}(S) > 0$, then $\dim_{\mathrm{P}}(R) \geq 1-\epsilon$. The reduction thus serves as a *randomness extractor* that increases the algorithmic randomness of S, as measured by constructive dimension.

A number of applications of this result shed new light on the constructive dimensions of wtt degrees (and, by extension, Turing degrees). A lower bound of $\dim_{\mathrm{H}}(S)/\dim_{\mathrm{P}}(S)$ is shown to hold for the wtt degree of any sequence S. A new proof is given of a previously-known zero-one law for the constructive packing dimension of wtt degrees. It is also shown that, for any *regular* sequence S (that is, $\dim_{\mathrm{H}}(S) = \dim_{\mathrm{P}}(S)$) such that $\dim_{\mathrm{H}}(S) > 0$, the wtt degree of S has constructive Hausdorff and packing dimension equal to 1.

Finally, it is shown that no single Turing reduction can be a *universal* constructive Hausdorff dimension extractor.

Keywords: constructive dimension, weak truth-table, extractor, degree, randomness

1 Introduction

Hausdorff [5] initiated the study of dimension as a general framework to define the size of subsets of metric spaces. Recently this framework had been effectivized; Lutz [9] gives an overview of this historical development. Furthermore, Lutz [8, Section 6] reviews early results that anticipated the effectivization of

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Hausdorff dimension. *Constructive Hausdorff dimension* was defined by Lutz [8] to study effective dimension at the level of computability theory. Intuitively, given an infinite binary sequence S – interpreted as a language or decision problem – the constructive Hausdorff dimension $\dim_{\mathrm{H}}(S)$ of S is a real number in the interval [0,1] indicating the density of algorithmic randomness of the sequence. The constructive Hausdorff dimension of a class $\mathcal{C}$ of sequences is the supremum of the dimensions of individual sequences in $\mathcal{C}$. For many classes $\mathcal{C}$ of interest in computability theory, the problem of determining the constructive Hausdorff dimension of $\mathcal{C}$ remains open.

Reimann [14] investigated in particular whether there are degrees of fractional constructive Hausdorff dimension. Stated in terms of individual sequences, Reimann asked which reducibilities (such as Turing, many-one, weak truth-table, etc.) are capable of increasing the constructive Hausdorff dimension of a sequence. We call such a reduction a *dimension extractor*, since its purpose bears a resemblance to that of the *randomness extractors* of computational complexity [18], which are algorithms that turn a source of weak randomness (a probabilistic source with low entropy) into a source of strong randomness (a source with high entropy). Viewing a sequence with positive, but still fractional, constructive Hausdorff dimension as a weak source of randomness, Reimann essentially asked whether such randomness can be extracted via a reduction to create a sequence with dimension closer to 1. If such extraction is *not* possible for some sequence S, this indicates that the degree of S under the reduction has fractional dimension.

A number of negative results for dimension extractors are known. Reimann and Terwijn [14, Theorem 3.10] proved that there are many-one and bounded truth-table degrees with constructive Hausdorff dimension strictly between 0 and 1. Later Reimann and Slaman [15] extended this result to truth-table degrees. Stephan [20] showed that there is a relativized world in which there exists a wtt degree of constructive Hausdorff dimension between $\frac{1}{4}$ and $\frac{1}{2}$. Furthermore, Nies and Reimann [11] obtained a non-relativized variant of this result and constructed, for each rational α between 0 and 1, a wtt degree of constructive Hausdorff dimension α.

Doty [3] attempted positive results by considering the interaction between constructive Hausdorff dimension and *constructive packing dimension* [1], a dual quantity that is a constructive effectivization of classical packing dimension [21, 22], another widely-studied fractal dimension. The constructive packing dimension $\dim_{\mathrm{P}}(S)$ of a sequence S always obeys

$$0 \leq \dim_{\mathrm{H}}(S) \leq \dim_{\mathrm{P}}(S) \leq 1,$$

with each inequality tight in the strong sense that there are sequences S in which $\dim_{\mathrm{H}}(S)$ and $\dim_{\mathrm{P}}(S)$ may take on any values obeying the stated constraint. Doty showed that every sequence S with $\dim_{\mathrm{H}}(S) > 0$ is Turing equivalent to a sequence R with $\dim_{\mathrm{P}}(R) \geq 1 - \epsilon$, for arbitrary $\epsilon > 0$. This implies that the constructive packing dimension of the Turing degree of any sequence S with $\dim_{\mathrm{H}}(S) > 0$ is equal to 1. Unfortunately, since $\dim_{\mathrm{H}}(R) \leq \dim_{\mathrm{P}}(R)$, this Turing reduction constitutes a weaker example of a dimension extractor than that sought by Reimann and it tells us nothing of the constructive dimensions of arbitrary Turing degrees.

We obtain in the current paper stronger positive results for constructive dimension extractors. Our main result, in section 2, is that, given any infinite sequence S and $\epsilon > 0$, there exists $R \equiv_{\mathrm{wtt}} S$ such that $\dim_{\mathrm{H}}(R) \geq \frac{\dim_{\mathrm{H}}(S)}{\dim_{\mathrm{P}}(S)} - \epsilon$ and, if $\dim_{\mathrm{P}}(S) > 0$, then $\dim_{\mathrm{P}}(R) \geq 1 - \epsilon$. This has immediate consequences for the dimensions of weak truth-table degrees:

- Given any sequence S, $\dim_{\mathrm{H}}(\deg_{\mathrm{wtt}}(S)) \geq \frac{\dim_{\mathrm{H}}(S)}{\dim_{\mathrm{P}}(S)}$.
- If $\dim_{\mathrm{P}}(S) > 0$, then $\dim_{\mathrm{P}}(\deg_{\mathrm{wtt}}(S)) = 1$, implying that *every* wtt degree has constructive packing dimension 0 or 1.
- Given any *regular* sequence S such that $\dim_{\mathrm{H}}(S) > 0$, $\dim_{\mathrm{H}}(\deg_{\mathrm{wtt}}(S)) = 1$, where a sequence S is regular if it satisfies $\dim_{\mathrm{H}}(S) = \dim_{\mathrm{P}}(S)$.

In section 3, we use Theorem 2.1 to show that, for every $\alpha > 0$, there is no *universal* Turing reduction that is guaranteed to extract dimension from all sequences of dimension at least α.

Before going into the details of the results, we introduce the concepts and notations formally.

Notation. We refer the reader to the textbooks of Li and Vitányi [6] for an introduction to Kolmogorov complexity and algorithmic information theory and of Odifreddi [13] and Soare [19] for an introduction to computability theory. Although we follow mainly the notation in these books, we nevertheless want to remind the reader on the following definitions, either for the readers' convenience or because we had to choose between several common ways of denoting the corresponding mathematical objects.

All logarithms are base 2. $\mathbb{N}$ denotes the set $\{0, 1, 2, 3, \ldots\}$ of the natural numbers including 0. $\{0,1\}^*$ denotes the set of all finite, binary *strings*. For all $x \in \{0,1\}^*$, $|x|$ denotes the *length* of x. λ denotes the empty string. $\mathbf{C} = \{0,1\}^\infty$ denotes the *Cantor space*, the set of all infinite, binary *sequences*. For $x \in \{0,1\}^*$ and $y \in \{0,1\}^* \cup \mathbf{C}$, xy denotes the concatenation of x and y, $x \sqsubseteq y$ denotes that x is a *prefix* of y (that is, there exists $u \in \{0,1\}^* \cup \mathbf{C}$ such that $xu = y$) and $x \sqsubset y$ denotes that $x \sqsubseteq y$ and $x \neq y$. For $S \in \{0,1\}^* \cup \mathbf{C}$ and $i, j \in \mathbb{N}$, $S[i]$ denotes the i^{th} bit of S, with $S[0]$ being the leftmost bit, $S[i \,..\, j]$ denotes the substring consisting of the i^{th} through j^{th} bits of S (inclusive), with $S[i \,..\, j] = \lambda$ if $i > j$.

Reductions and Compression. Let M be a Turing machine and $S \in \mathbf{C}$. We say M *computes* S if M on input $n \in \mathbb{N}$ (written $M(n)$), outputs the string $S[0 \,..\, n-1]$. We define an *oracle Turing machine* to be a Turing machine M that can make constant-time queries to an oracle sequence and we let OTM denote the set of all oracle Turing machines. For $R \in \mathbf{C}$, we say M operates *with oracle* R if, whenever M makes a query to index $n \in \mathbb{N}$, the bit $R[n]$ is returned. We write M^R to denote the oracle Turing machine M with oracle R.

Let $S, R \in \mathbf{C}$ and $M \in$ OTM. We say S *is Turing reducible to* R *via* M and we write $S \leq_{\mathrm{T}} R$ *via* M, if M^R computes S (that is, if $M^R(n) = S[0 \,..\, n-1]$

for all $n \in \mathbb{N}$). In this case, write $R = M(S)$. We say S *is Turing reducible to* R and we write $S \leq_{\mathrm{T}} R$, if there exists $M \in \mathrm{OTM}$ such that $S \leq_{\mathrm{T}} R$ via M. We say S is *Turing equivalent* to R, and we write $S \equiv_{\mathrm{T}} R$, if $S \leq_{\mathrm{T}} R$ and $R \leq_{\mathrm{T}} S$. The *Turing lower span* of S is $\mathrm{span}_{\mathrm{T}}(S) = \{ R \in \mathbf{C} \mid R \leq_{\mathrm{T}} S \}$ and the *Turing degree* of S is $\deg_{\mathrm{T}}(S) = \{ R \in \mathbf{C} \mid R \equiv_{\mathrm{T}} S \}$.

Let $S, R \in \mathbf{C}$ and $M \in \mathrm{OTM}$ such that $S \leq_{\mathrm{T}} R$ via M. Let the notion $\#(M^R, S[0\,..\,n{-}1])$ denote the *query usage of* M^R *on* $S[0\,..\,n{-}1]$, the number of bits of R queried by M when computing the string $S[0\,..\,n{-}1]$.[4] We say S *is weak truth-table (wtt) reducible to* R *via* M and we write $S \leq_{\mathrm{wtt}} R$ *via* M, if $S \leq_{\mathrm{T}} R$ via M and there is a computable function $q : \mathbb{N} \to \mathbb{N}$ such that, for all $n \in \mathbb{N}$, $\#(M^R, S[0\,..\,n{-}1]) \leq q(n)$. Define $S \leq_{\mathrm{wtt}} R$, $S \equiv_{\mathrm{wtt}} R$, $\mathrm{span}_{\mathrm{wtt}}(S)$ and $\deg_{\mathrm{wtt}}(S)$ analogously to their counterparts for Turing reductions. Define

$$\rho_M^-(S, R) = \liminf_{n\to\infty} \frac{\#(M^R, S[0\,..\,n{-}1])}{n},$$
$$\rho_M^+(S, R) = \limsup_{n\to\infty} \frac{\#(M^R, S[0\,..\,n{-}1])}{n}.$$

Viewing R as a compressed version of S, $\rho_M^-(S, R)$ and $\rho_M^+(S, R)$ are respectively the best- and worst-case compression ratios as M decompresses R into S. Note that $0 \leq \rho_M^-(S, R) \leq \rho_M^+(S, R) \leq \infty$.

The following lemma is useful when one wants to compose two reductions:

Lemma 1.1. [2] *Let* $S, S', S'' \in \mathbf{C}$ *and* $M_1, M_2 \in \mathrm{OTM}$ *such that* $S' \leq_{\mathrm{T}} S$ *via* M_1 *and* $S'' \leq_{\mathrm{T}} S'$ *via* M_2. *There exists* $M \in \mathrm{OTM}$ *such that* $S'' \leq_{\mathrm{T}} S$ *via* M *and:*

$$\rho_M^+(S'', S) \leq \rho_{M_2}^+(S'', S')\rho_{M_1}^+(S', S).$$
$$\rho_M^-(S'', S) \leq \rho_{M_2}^-(S'', S')\rho_{M_1}^+(S', S).$$
$$\rho_M^-(S'', S) \leq \rho_{M_2}^+(S'', S')\rho_{M_1}^-(S', S).$$

(The last bound is not explicitly stated in [2], but it holds for the same reason as the second one).

For $S \in \mathbf{C}$, the *lower and upper Turing compression ratios of* S are respectively defined as

$$\rho^-(S) = \min_{\substack{R\in\mathbf{C}\\ M\in\mathrm{OTM}}} \left\{ \rho_M^-(S, R) \;\middle|\; S \leq_{\mathrm{T}} R \text{ via } M \right\},$$
$$\rho^+(S) = \min_{\substack{R\in\mathbf{C}\\ M\in\mathrm{OTM}}} \left\{ \rho_M^+(S, R) \;\middle|\; S \leq_{\mathrm{T}} R \text{ via } M \right\}.$$

Doty [2] showed that the above minima exist. Note that $0 \leq \rho^-(S) \leq \rho^+(S) \leq 1$.

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[4] If we instead define $\#(M^R, S[0\,..\,n{-}1])$ to be the index of the rightmost bit of R queried by M when computing $S[0\,..\,n{-}1]$, all results of the present paper still hold.

Constructive Dimension. Lutz [8] gives an introduction to the theory of constructive dimension. We use Mayordomo's characterization [10] of the constructive dimensions of sequences. For all $S \in \mathbf{C}$, the *constructive Hausdorff dimension* and the *constructive packing dimension* of S are respectively defined as

$$\dim_{\mathrm{H}}(S) = \liminf_{n\to\infty} \frac{\mathrm{C}(S[0\,..\,n{-}1])}{n} \text{ and } \dim_{\mathrm{P}}(S) = \limsup_{n\to\infty} \frac{\mathrm{C}(S[0\,..\,n{-}1])}{n},$$

where $\mathrm{C}(w)$ denotes the *Kolmogorov complexity* of $w \in \{0,1\}^*$ (see [6]). If $\dim_{\mathrm{H}}(S) = \dim_{\mathrm{P}}(S)$, we say S is a *regular* sequence. Doty [2] showed that, for all $S \in \mathbf{C}$, $\rho^-(S) = \dim_{\mathrm{H}}(S)$ and $\rho^+(S) = \dim_{\mathrm{P}}(S)$.

For all $X \subseteq \mathbf{C}$, the *constructive Hausdorff dimension* and the *constructive packing dimension* of X are respectively defined as

$$\dim_{\mathrm{H}}(X) = \sup_{S\in X} \dim_{\mathrm{H}}(S) \text{ and } \dim_{\mathrm{P}}(X) = \sup_{S\in X} \dim_{\mathrm{P}}(S).$$

2 Constructive Dimension Extractors

Nies and Reimann [11] showed that wtt reductions cannot always extract constructive dimension.

Theorem 2.1 (Nies and Reimann [11]). *For every rational number α with $0 < \alpha < 1$, there exists a sequence $S \in \mathbf{C}$ such that, for all wtt reductions M, $\dim_{\mathrm{H}}(M(S)) \leq \dim_{\mathrm{H}}(S) = \alpha$.*

Ryabko [16, 17] discovered the next theorem.

Theorem 2.2 (Ryabko [16, 17]). *For all $S \in \mathbf{C}$ and $\delta > 0$, there exists $R \in \mathbf{C}$ and $N_d \in \mathrm{OTM}$ such that*

1. *$S \leq_{\mathrm{T}} R$ via N_d and $R \leq_{\mathrm{T}} S$.*
2. *$\rho^-_{N_d}(S, R) \leq \dim_{\mathrm{H}}(S) + \delta$.*

The following theorem was shown in [2].

Theorem 2.3 (Doty [2]). *There is an oracle Turing machine M_d such that, for all $S \in \mathbf{C}$, there exists $R \in \mathbf{C}$ such that*

1. *$S \leq_{\mathrm{wtt}} R$ via M_d.*
2. *$\rho^-_{M_d}(S, R) = \dim_{\mathrm{H}}(S)$.*
3. *$\rho^+_{M_d}(S, R) = \dim_{\mathrm{P}}(S)$.*

The following theorem, which is similar to Ryabko's Theorem 2.2, shows that the decoding machine M_d of Theorem 2.3 can also be reversed if the compression requirements are weakened.

Theorem 2.4. *Let M_d be the oracle Turing machine from Theorem 2.3. For all $\delta > 0$, there is an oracle Turing machine M_e such that, for all $S \in \mathbf{C}$, there is a sequence $R' \in \mathbf{C}$ such that*

1. $S \leq_{\mathrm{wtt}} R'$ *via* M_d *and* $R' \leq_{\mathrm{wtt}} S$ *via* M_e.
2. $\rho^{+}_{M_d}(S, R') \leq \dim_{\mathrm{P}}(S) + \delta$.

Proof. Let $S \in \mathbf{C}$ and choose R for S as in Theorem 2.3. Let $\delta > 0$ and let $D \in (\dim_{\mathrm{P}}(S), \dim_{\mathrm{P}}(S) + \delta)$ be rational. By Theorem 2.3, there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$, $\#(M_d^R, S[0\,..\,n-1]) < Dn$.

M_e will make use of the oracle Turing machine M_d. The proof of Theorem 2.3 in [2] shows that M_d has the following useful properties. First, write $S = s_1 s_2 s_3 \ldots$ and $R = r_1 r_2 r_3 \ldots$, where each $s_i, r_i \in \{0,1\}^*$ are blocks such that $|s_i| = i$ and $|r_i| \leq |s_i| + o(|s_i|)$.

- M_d computes S from R in stages, where it outputs the block s_i on the i^{th} stage.
- Assuming that M_d has already computed $s_1 \ldots s_i$, M_d uses only the block r_{i+1} and the prefix $s_1 \ldots s_i$ to compute s_{i+1}.

Because of these properties, we can use M_d to search for a sequence R' that satisfies requirements 1 and 2 in the statement of Theorem 2.4. By Theorem 2.3, R satisfies these requirements, so such an R' will exist. By the above two properties of M_d, if we find a string $r' = r'_1 \ldots r'_i$ that satisfies requirements 1 and 2 (in the sense described below), we will always be able to find an extension $r'' = r'_{i+1} \ldots r'_j$ (for some $j > i$) such that $r'r''$ continues to satisfy the requirements. It will not matter if $r' \not\sqsubseteq R$, since M_d does not use the portion of R coming before block r_{i+1} to compute s_{i+1}. In other words, to reverse the computation of $M_d^{R'}$ and compute R' from S, we don't need to find the R from Theorem 2.3; we need only to find an R' that is "close enough".

Define the oracle Turing machine M_e with oracle $S \in \mathbf{C}$ as follows. Let $i \in \mathbb{N}$ and assume inductively that the prefix $r' = r'_1 \ldots r'_i \sqsubset R'$ has been computed, so that, letting $|s_1 \ldots s_i| = n$,

(a) $M_d^{r'}(n)$ outputs $S[0\,..\,n-1]$,
(b) for all k with $n_0 \leq k \leq n$, $\#(M_d^{r'}, S[0\,..\,k-1]) \leq Dk$.

Let N be the smallest integer greater than 2^n such that $S[0\,..\,N-1] = s_1 \ldots s_{i'}$, for some $i' \in \mathbb{N}$. M_e^S searches all strings $r'' \in \{0,1\}^N$ until it finds one that satisfies

(a) $M_d^{r'r''}(N)$ outputs $S[0\,..\,N-1]$,
(b) for all k with $n_0 \leq k \leq N$, $\#(M_d^{r'r''}, S[0\,..\,k-1]) \leq Dk$.

M_e^S then outputs r'' and saves it for the computation of the next extension of R'. By the existence of R from Theorem 2.3 and a simple induction on the stages of computation that M_e performs and the fact that N is asymptotically larger than n, M_e^S will always be able to find such an r''. Therefore, in the output sequence R', for all but finitely many N, requirement (b) will be satisfied. Therefore the sequence R' will satisfy the two requirements of Theorem 2.4.

Finally, for any n, $M_e(n)$ makes no more than 2^{2n} queries to S and therefore M_e computes a wtt reduction. $\square$

The following theorem is the main result of this paper. It states that constructive packing dimension can be almost optimally extracted from a sequence of positive packing dimension, while at the same time, constructive Hausdorff dimension is *partially* extracted from this sequence, if it has positive Hausdorff dimension and packing dimension less than 1. The machine M_e from Theorem 2.4 serves as the extractor. Intuitively, this works because M_e compresses the sequence S into the sequence R. Since R is a compressed representation of S, R must itself be more incompressible than S. However, because dimension measures the compressibility of a sequence, this means that the constructive dimensions R are greater than those of S.

Theorem 2.5. *For all $\epsilon > 0$ and $S \in \mathbf{C}$ such that $\dim_{\mathrm{P}}(S) > 0$, there exists $R \equiv_{\mathrm{wtt}} S$ such that $\dim_{\mathrm{P}}(R) \geq 1 - \epsilon$ and $\dim_{\mathrm{H}}(R) \geq \frac{\dim_{\mathrm{H}}(S)}{\dim_{\mathrm{P}}(S)} - \epsilon$.*

Proof. Let $\epsilon > 0$ and $S \in \mathbf{C}$ such that $\dim_{\mathrm{P}}(S) > 0$. Let $\delta > 0$ and R', M_d be as in Theorem 2.4. Let $R'' \in \mathbf{C}$ and $M \in \mathrm{OTM}$ such that $R' \leq_{\mathrm{T}} R''$ via M, $\rho^-_M(R', R'') = \dim_{\mathrm{H}}(R')$ and $\rho^+_M(R', R'') = \dim_{\mathrm{P}}(R')$ (the existence of M and R'' is asserted by Theorem 2.3). By Lemma 1.1, we have

$$\rho^+(S) \leq \rho^+_{M_d}(S, R')\rho^+_M(R', R''),$$

which, by construction of R' and R'' implies $\rho^+(S) \leq (\dim_{\mathrm{P}}(S) + \delta) \dim_{\mathrm{P}}(R')$. Since $\rho^+(S) = \dim_{\mathrm{P}}(S)$,

$$\dim_{\mathrm{P}}(R') \geq \frac{\dim_{\mathrm{P}}(S)}{\dim_{\mathrm{P}}(S) + \delta}.$$

Moreover (by Lemma 1.1 again), $\rho^-(S) \leq \rho^+_{M_d}(S, R')\rho^-_M(R', R'')$, which, by construction of R' and R'', implies $\rho^-(S) \leq (\dim_{\mathrm{P}}(S) + \delta) \dim_{\mathrm{H}}(R')$. Since $\rho^-(S) = \dim_{\mathrm{H}}(S)$,

$$\dim_{\mathrm{H}}(R') \geq \frac{\dim_{\mathrm{H}}(S)}{\dim_{\mathrm{P}}(S) + \delta}.$$

Taking δ small enough, we get by the above inequalities: $\dim_{\mathrm{P}}(R) \geq 1 - \epsilon$ and $\dim_{\mathrm{H}}(R) \geq \frac{\dim_{\mathrm{H}}(S)}{\dim_{\mathrm{P}}(S)} - \epsilon$. □

Theorem 2.5 has a number of applications, stated in the following corollaries, which shed light on the constructive dimensions of sequences, spans and degrees.

Corollary 2.6. *Let $S \in \mathbf{C}$ and assume that $\dim_{\mathrm{H}}(S) > 0$. Then $\dim_{\mathrm{H}}(\mathrm{deg}_{\mathrm{T}}(S))$, $\dim_{\mathrm{H}}(\mathrm{deg}_{\mathrm{wtt}}(S))$, $\dim_{\mathrm{H}}(\mathrm{span}_{\mathrm{T}}(S))$ and $\dim_{\mathrm{H}}(\mathrm{span}_{\mathrm{wtt}}(S))$ are all at least $\frac{\dim_{\mathrm{H}}(S)}{\dim_{\mathrm{P}}(S)}$.*

We obtain a zero-one law for the constructive packing dimension of Turing and weak truth-table lower spans and degrees.

Corollary 2.7. *For all $S \in \mathbf{C}$, the dimensions $\dim_{\mathrm{P}}(\mathrm{deg}_{\mathrm{T}}(S))$, $\dim_{\mathrm{P}}(\mathrm{span}_{\mathrm{T}}(S))$, $\dim_{\mathrm{P}}(\mathrm{deg}_{\mathrm{wtt}}(S))$ and $\dim_{\mathrm{P}}(\mathrm{span}_{\mathrm{wtt}}(S))$ are each either 0 or 1.*

Therefore Theorem 2.1, establishing the existence of wtt degrees of fractional constructive Hausdorff dimension, does not extend to constructive packing dimension. Because of Theorem 2.1, we must settle for more conditional results for constructive Hausdorff dimension. We focus attention on regular sequences.

Corollary 2.8. *For all* $\epsilon > 0$ *and all regular* $S \in \mathbf{C}$ *such that* $\dim_{\mathrm{H}}(S) > 0$*, there exists* $R \equiv_{\mathrm{wtt}} S$ *such that* $\dim_{\mathrm{H}}(R) \geq 1 - \epsilon$*.*

Corollary 2.9. *For all regular* $S \in \mathbf{C}$ *such that* $\dim_{\mathrm{H}}(S) > 0$*,*

$$\dim_{\mathrm{H}}(\mathrm{span}_{\mathrm{wtt}}(S)) = \dim_{\mathrm{H}}(\deg_{\mathrm{wtt}}(S)) = \dim_{\mathrm{H}}(\mathrm{span}_{\mathrm{T}}(S)) = \dim_{\mathrm{H}}(\deg_{\mathrm{T}}(S)) =$$
$$\dim_{\mathrm{P}}(\mathrm{span}_{\mathrm{wtt}}(S)) = \dim_{\mathrm{P}}(\deg_{\mathrm{wtt}}(S)) = \dim_{\mathrm{P}}(\mathrm{span}_{\mathrm{T}}(S)) = \dim_{\mathrm{P}}(\deg_{\mathrm{T}}(S)) = 1.$$

It remains open whether every Turing lower span or degree of positive constructive Hausdorff dimension contains a regular sequence of positive constructive Hausdorff dimension. If so, this would imply a zero-one law for constructive Hausdorff dimension similar to Corollary 2.7.

We note that the zero-one law for the constructive packing dimension of Turing and wtt lower spans and degrees also follows from the following theorem due to Fortnow, Hitchcock, Pavan, Vinodchandran and Wang [4], giving a polynomial-time extractor for constructive packing dimension. For $R, S \in \mathbf{C}$, write $R \leq_{\mathrm{T}}^{\mathrm{p}} S$ if $R \leq_{\mathrm{T}} S$ via an OTM that, on input n, runs in time polynomial in n, and similarly for $\equiv_{\mathrm{T}}^{\mathrm{p}}$.

Theorem 2.10 ([4]). *For all* $\epsilon > 0$ *and all* $S \in \mathbf{C}$ *such that* $\dim_{\mathrm{P}}(S) > 0$*, there exists* $R \equiv_{\mathrm{T}}^{\mathrm{p}} S$ *such that* $\dim_{\mathrm{P}}(R) \geq 1 - \epsilon$*.*

In fact, Theorem 2.10 holds for any resource-bounded packing dimension [7] defined by Turing machines allowed at least polynomial space, which includes constructive packing dimension as a special case, thus proving a spectrum of zero-one packing dimension laws for various dimensions above polynomial space and degrees and lower spans that are at least polynomial-time computable.

3 Nonexistence of Universal Extractors

The wtt reduction in the proof of Theorem 2.5 is uniform in the sense that, for all $\epsilon > 0$, there is a *single* wtt reduction M, universal for ϵ and all sequences S, such that $\dim_{\mathrm{H}}(M(S)) \geq \dim_{\mathrm{H}}(S)/\dim_{\mathrm{P}}(S) - \epsilon$.

While it remains open whether Turing reductions can extract constructive Hausdorff dimension, we can show that there is no *universal* Turing reduction that is guaranteed to increase – to a fixed amount – the dimension of all sequences of sufficiently large dimension.

Theorem 3.1. *For every Turing reduction* M *and all reals* α, β *with* $0 < \alpha < \beta < 1$*, there exists* $S \in \mathbf{C}$ *with* $\dim_{\mathrm{H}}(S) \geq \alpha$ *such that* $M(S)$ *does not exist or* $\dim_{\mathrm{H}}(M(S)) < \beta$*.*

Proof. For this proof, it will be convenient to say that $R \leq_{\mathrm{T}} S$ via M if $M^S(n)$ outputs $R[n]$, rather than $R[0\,..\,n-1]$, bearing in mind that both definitions of a Turing reduction are equivalent.

Suppose for the sake of contradiction that there exist reals α, β with $0 < \alpha < \beta < 1$ and a Turing reduction M such that, for all $S \in \mathbf{C}$ satisfying $\dim_{\mathrm{H}}(S) \geq \alpha$, then $\dim_{\mathrm{H}}(R) \geq \beta$, where $R = M(S)$. Fix rationals α', γ such that $\alpha < \alpha' < \gamma < \beta$. We will convert M into a truth-table reduction N (a reduction that halts on all oracles, which is also a wtt reduction) that guarantees the slightly weaker condition that if $\dim_{\mathrm{H}}(S) > \alpha'$, then $\dim_{\mathrm{H}}(N(S)) \geq \beta$. Then for any $S \in \mathbf{C}$ such that $\dim_{\mathrm{H}}(S) = \gamma > \alpha'$, it follows that $\dim_{\mathrm{H}}(N(S)) \geq \beta > \gamma = \dim_{\mathrm{H}}(S)$, which contradicts Theorem 2.1.

On input $n \in \mathbb{N}$ and with oracle sequence S, $N^S(n)$ simulates $M^S(n)$. In parallel, for all integers $m > n$, N searches for a program of length at most $\alpha' m$ computing $S[0\,..\,m-1]$. If N finds such a program before the simulation of $M^S(n)$ terminates, then N outputs 0. If instead the simulation of $M^S(n)$ halts before such a short program is found, then N outputs $R[n]$, the output bit of $M^S(n)$.

If $\dim_{\mathrm{H}}(S) < \alpha'$, then for infinitely many $m \in \mathbb{N}$, $\mathrm{C}(S[0\,..\,m-1]) \leq \alpha' m$. Therefore N^S halts, although the output sequence $N(S)$ may contain a lot of 0's, which is acceptable because we do not care what N outputs if $\dim_{\mathrm{H}}(S) < \alpha'$.

If $\dim_{\mathrm{H}}(S) \geq \alpha'$, then M^S is guaranteed to halt and to compute R such that $\dim_{\mathrm{H}}(R) \geq \beta$. Therefore N^S halts. If $\dim_{\mathrm{H}}(S) = \alpha'$, then once again, we do not care what N outputs. If $\dim_{\mathrm{H}}(S) > \alpha'$, then only finitely many m satisfy $\mathrm{C}(S[0\,..\,m-1]) \leq \alpha' m$. Therefore the parallel search for short programs will never succeed once N begins checking only prefixes of S of sufficiently large length. This means that from that point on, N will simulate M exactly, computing a sequence R' that is a finite variation of R. Since dimension is unchanged under finite variations, $\dim_{\mathrm{H}}(R') = \dim_{\mathrm{H}}(R) \geq \beta$. □

Theorem 3.1 tells us that, contrary to the proofs of Theorems 2.4 and 2.5, any extractor construction for Turing reductions must make use of some property of the sequence beyond a simple bound on its dimension.

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