Limitations of Self-Assembly at Temperature 1*

David Doty†, Matthew J. Patitz, and Scott M. Summers‡

Department of Computer Science
Iowa State University
Ames, IA 50011, USA
{ddoty,mpatitz,summers}@cs.iastate.edu

Abstract

We prove that if a set $X \subseteq \mathbb{Z}^2$ weakly self-assembles at temperature 1 in a deterministic (Winfree) tile assembly system satisfying a natural condition known as *pumpability*, then X is a semilinear set. This shows that only the most simple of infinite shapes and patterns can be constructed using pumpable temperature 1 tile assembly systems, and gives evidence for the thesis that temperature 2 or higher is required to carry out general-purpose computation in a deterministic two-dimensional tile assembly system. We employ this result to show that, unlike the case of temperature 2 self-assembly, no discrete self-similar fractal weakly self-assembles at temperature 1 in a pumpable tile assembly system.

1 Introduction

Self-assembly is a bottom-up process by which a small number of fundamental components automatically coalesce to form a target structure. In 1998, Winfree [19] introduced the (abstract) Tile Assembly Model (TAM) – an "effectivization" of Wang tiling [17,18] – as an over-simplified mathematical model of the DNA self-assembly pioneered by Seeman [15]. In the TAM, the fundamental components are un-rotatable, but translatable square "tile types" whose sides are labeled with glue "colors" and "strengths." Two tiles that are placed next to each other *interact* if the glue colors on their abutting sides match, and they *bind* if the strength on their abutting sides matches with total strength at least a certain ambient "temperature," usually taken to be 1 or 2 (i.e., a single strength 2 bond, or two strength 1 bonds).

Despite its deliberate over-simplification, the TAM is a computationally and geometrically expressive model at temperature 2. The reason is that, at temperature 2, certain tiles are not permitted to bond until *two* tiles are already present to match the glues on the bonding sides, which enables cooperation between different tile types to control the placement of new tiles. Winfree [19] proved that at temperature 2 the TAM is computationally universal and thus can be directed algorithmically.

*This research was supported in part by National Science Foundation grants 0652569 and 0728806.

†This author's research was partially supported by NSF grant CCF:0430807, by Natural Sciences and Engineering Research Council of Canada (NSERC) Discovery Grant R2824A01 and the Canada Research Chair Award in Biocomputing, and by the NSF Computing Innovation Fellowship grant.

‡This author's research was supported in part by NSF-IGERT Training Project in Computational Molecular Biology Grant number DGE-0504304.

In contrast, we aim to prove that the lack of cooperation at temperature 1 inhibits the sort of complex behavior possible at temperature 2. Our main theorem concerns the *weak self-assembly* of subsets of $\mathbb{Z}^2$ using temperature 1 tile assembly systems. Informally, a set $X \subseteq \mathbb{Z}^2$ weakly self-assembles in a tile assembly system $\mathcal{T}$ if some of the tile types of $\mathcal{T}$ are painted black, and $\mathcal{T}$ is guaranteed to self-assemble into an assembly α of tiles such that X is precisely the set of integer lattice points at which α contains black tile types. As an example, Winfree [19] exhibited a temperature 2 tile assembly system, directed by a clever XOR-like algorithm, that weakly self-assembles a well-known set, the discrete Sierpinski triangle, onto the first quadrant. Note that the underlying *shape* (set of all points containing a tile, whether black or not) of Winfree's construction is an infinite canvas that covers the entire first quadrant, onto which a more sophisticated set, the discrete Sierpinski triangle, is painted.

We study only *directed* tile systems in this paper, meaning those tile systems that produce a unique terminal assembly, where an assembly is defined not only by its shape (which points have a tile), but also by which tile types appear at which points within the shape. We show that under a natural assumption, directed temperature 1 tile systems weakly self-assemble only a limited class of sets. To prove our main result, we require the hypothesis that the tile system is *pumpable*. Informally, this means that every sufficiently long path of tiles in an assembly of this system contains a segment in which the same tile type repeats (a condition clearly implied by the pigeonhole principle), and that furthermore, the subpath between these two occurrences can be repeated indefinitely ("pumped") along the same direction as the first occurrence of the segment, without "colliding" with a previous portion of the path. We give an example in Section 3 (Figure 1) of a path in which the same tile type appears twice, yet the segment between the appearances cannot be pumped without eventually resulting in a collision that prevents additional pumping. The hypothesis of pumpability states (roughly) that in every sufficiently long path, despite the presence of some repeating tiles that cannot be pumped, *there exists* a segment in which the same tile type repeats that *can* be pumped. In the above-mentioned example, the paths constructed to create a blocked segment always contain some previous segment that *is* pumpable. We conjecture that this phenomenon, pumpability, occurs in every temperature 1 tile assembly system that produces a unique infinite structure (i.e., is deterministic). We discuss this conjecture in greater detail in Section 6. We also prove that if the terminal assembly of a directed tile assembly system has no glue mismatches between neighboring tiles, then the tile assembly system is pumpable.

A *linear* set $X \subseteq \mathbb{Z}^2$ is a set of integer lattice points with the property that there are three vectors $\vec{b}$ (the "base point" of the set), $\vec{u}$, and $\vec{v}$ (the two periods of the set), such that $X = \left\{ \left. \vec{b} + n \cdot \vec{u} + m \cdot \vec{v} \; \right| \; n, m \in \mathbb{N} \right\}$. That is, a linear set is a "two-dimensionally periodic" set that repeats infinitely along two vectors (linearly independent vectors in the non-degenerate case), starting at some base point $\vec{b}$. We show that any directed, pumpable, temperature 1 tile assembly system weakly self-assembles a set $X \subseteq \mathbb{Z}^2$ that is a finite union of linear sets, known also as a *semilinear* set. Semilinear sets are computationally very simple. They have been shown, for example, to be equivalent to those sets definable in the (very weak) theory of Presburger arithmetic [11,16], the first-order theory of natural numbers with only the addition operation (but lacking multiplication). Therefore such sets are *logically* very weak. They have also been shown [5] to characterize the set of languages (of tuples of natural numbers) generated by *reversal-bounded counter machines*, which are finite-state machines without an input tape, and a finite number (two in our case since we work in $\mathbb{Z}^2$) of counters – a register containing a natural number that can be incremented, decremented, or tested for 0 – with the property that each counter can only switch between incrementing and decrementing a bounded number of times. For completeness, we give a self-contained proof (Observation 4.2) that the projection of any semilinear set onto the x-axis, when expressed in unary, is a regular language. This observation and [5] imply that semilinear sets are *computationally* very weak.

It is our contention that weak self-assembly captures the intuitive notion of what it means to "compute" with a tile assembly system. For example, the use of tile assembly systems to build shapes is captured by requiring all tile types to be black, in order to ask what set of integer lattice points contain any tile at all (so-called *strict self-assembly*). However, weak self-assembly is a more general notion. For example, Winfree's above mentioned result shows that the discrete Sierpinski triangle weakly self-assembles at temperature 2 [14], yet this shape does not strictly self-assemble at *any* temperature [7]. Hence weak self-assembly allows for a more relaxed notion of set building, in which intermediate space can be used for computation, without requiring that the space filled to carry out the computation also represent the final result of the computation.

As another example, there is a standard construction [19] by which a single-tape Turing machine may be simulated by a temperature 2 tile assembly system. Regardless of the semantics of the Turing machine (whether it decides a language, enumerates a language, computes a function, etc.), it is routine to represent the result of the computation by the weak self-assembly of some set. For example, Patitz and Summers [10] showed that for any decidable language $A \subseteq \mathbb{N}$, A's projection along the X-axis $\big($the set $\{\ (x,0) \in \mathbb{N}^2 \ \big|\ x \in A\ \}\big)$ weakly self-assembles in a temperature 2 tile assembly system. As another example, if a Turing machine computes a function $f : \mathbb{N} \to \mathbb{N}$, it is routine to design a tile assembly system based on Winfree's construction such that, if the seed assembly is used to encode the binary representation of a number $n \in \mathbb{N}$, then the tile assembly system weakly self-assembles the set

$$\left\{ (k,0) \in \mathbb{N}^2 \;\middle|\; \begin{array}{l} \text{the } k^{\text{th}} \text{ least significant bit of the} \\ \text{binary representation of } f(n) \text{ is } 1 \end{array} \right\}.$$

Our result is motivated by the thesis that if a tile assembly system can reasonably be said to "compute", then the result of this computation can be represented in a straightforward manner as a set $X \subseteq \mathbb{Z}^2$ that weakly self-assembles in the tile assembly system, or a closely related tile assembly system. Our examples above provide evidence for this thesis, although it is as informal and unprovable as the Church-Turing thesis. On the basis of this claim, and the triviality of semilinear sets (shown more formally in Observation 4.2), we conclude that our main result implies that directed, pumpable, temperature 1 tile assembly systems are incapable of general-purpose deterministic computation, without further relaxing the model.

Related Work. Maňuch, Stacho, and Stoll [8,9] have shown lower bound results for a restricted class of *finite* assemblies at temperature 1. In particular, they show that uniquely assembling a finite assembly of diameter n in the Manhattan metric requires n unique tile types if the assembly is required to have no glue mismatches between adjacent tiles. This has the corollary that uniquely assembling an $n \times n$ square with no mismatches requires $2n-1$ unique tile types, matching the upper bound shown by Rothemund and Winfree [13].

Universal computation at temperature 1 *is* possible by changing the underlying model of self-assembly. There is a different sense in which undirected temperature 1 tile assembly systems may be considered to achieve "universal computation". Adleman has devised a nondeterministic temperature 1 tile assembly system, cited as a personal communication in Rothemund's Ph.D.

thesis [12], which does something similar to the following (the exact construction described next is reconstructed from conversations and may differ in details from Adleman's original construction, although the ideas are implicit in [1]). It simulates a single-tape Turing machine in the standard way, by assembling a space-time diagram of the configuration history of the Turing machine, with the growth front of the assembly crawling back and forth across the representation of the tape at time t (the t^{th} row of the assembly), in order to construct the tape at time $t+1$. Each cell encodes a value consisting of a tape symbol, the boolean value "Is the tape head here?", and if the tape head is present, what the state is. In order to read the value of a cell, the tile assembly system nondeterministically guesses the value. Any nondeterministic choice failing to correctly predict the value terminates the growth of the assembly, while correctly guessing continues the growth. Hence, many assemblies will be produced, but there is a unique largest assembly (infinite if the computation of the Turing machine is infinite), which represents the result of the computation. However, this construction does not weakly self-assemble any set representing the result of the computation, as the definition of weak self-assembly, while not requiring the tile assembly system to be directed, does require a unique set to be weakly assembled. So the question of whether nondeterministic computation can achieve universal computation has been answered, but in a weaker form than via the mechanism of weak self-assembly.

In addition, a construction that was personally communicated by Matthew Cook (also mentioned briefly in [12]), which is based on Adleman's nondeterministic construction, establishes that the aTAM is computationally universal with respect to directed, temperature 1 tile assembly systems that place tiles in three spatial dimensions. In fact, Cook's construction only uses two integer planes, "stacked" one on top of the other, to simulate an arbitrary Turing machine. No such directed two-dimensional universality result is known at the time of this writing.

Finally, Cook, Fu, and Schweller [2], in addition to independently discovering Cook's result of 3D temperature-1 universality, have also shown how to construct a temperature-1 tile assembly system that will simulate an arbitrary bounded algorithm with high probability, using nondeterministic competition between equally concentrated (equimolar) tiles in the randomized self-assembly model (see [3, 6] for a general definition). More precisely, for each $t(n)$-time-bounded Turing machine M and each string x of length n, and each $\epsilon > 0$, there is a randomized equimolar tile assembly system that will simulate M on input x, with probability at least $1 - \epsilon$.

This paper is organized as follows. Section 3 introduces new definitions and concepts that are required in proving our main theorem. Section 4 states and proves our main theorem, and contains an observation with a self-contained proof justifying the suggestion that semilinear sets are computationally very simple, based on their relationship to regular languages. Section 5 shows an application of our theorem, showing that, unlike the case of temperature 2, no non-trivial discrete self-similar fractal – such as the discrete Sierpinski triangle – weakly self-assembles at temperature 1 in a directed, pumpable tile assembly system. Section 6 concludes the paper and discusses open questions.

2 The Abstract Tile Assembly Model

We work in the 2-dimensional discrete space $\mathbb{Z}^2$. Define the set $U_2 = \{(0,1),(1,0),(0,-1),(-1,0)\}$ to be the set of all *unit vectors*, i.e., vectors of length 1 in $\mathbb{Z}^2$. We write $[X]^2$ for the set of all 2-element subsets of a set X. All *graphs* here are undirected graphs, i.e., ordered pairs $G = (V, E)$, where V is the set of *vertices* and $E \subseteq [V]^2$ is the set of *edges*.

We now give a brief and intuitive sketch of the Tile Assembly Model that is adequate for reading this paper. More formal details and discussion may be found in [7,12,13,19]. Our notation is that of [7], which contains a self-contained introduction to the Tile Assembly Model for the reader unfamiliar with the model.

Intuitively, a tile type t is a unit square that can be translated, but not rotated, having a well-defined "side $\vec{u}$" for each $\vec{u} \in U_2$. Each side $\vec{u}$ of t has a "glue" of "color" $\text{col}_t(\vec{u})$ – a string over some fixed alphabet Σ – and "strength" $\text{str}_t(\vec{u})$ – a nonnegative integer – specified by its type t. Two tiles t and t' that are placed at the points $\vec{a}$ and $\vec{a}+\vec{u}$ respectively, *bind* with *strength* $\text{str}_t(\vec{u})$ if and only if $(\text{col}_t(\vec{u}), \text{str}_t(\vec{u})) = (\text{col}_{t'}(-\vec{u}), \text{str}_{t'}(-\vec{u}))$.

Given a set T of tile types, an *assembly* is a partial function $\alpha : \mathbb{Z}^2 \dashrightarrow T$, with points $\vec{x} \in \mathbb{Z}^2$ at which $\alpha(\vec{x})$ is undefined interpreted to be empty space, so that dom α is the set of points with tiles. α is *finite* if $|\text{dom }\alpha|$ is finite. For assemblies α and α', we say that α is a *subconfiguration* of α', and write $\alpha \sqsubseteq \alpha'$, if $\text{dom }\alpha \subseteq \text{dom }\alpha'$ and $\alpha(\vec{x}) = \alpha'(\vec{x})$ for all $x \in \text{dom }\alpha$.

Let α be an assembly and $B \subseteq \mathbb{Z}^2$. *α restricted to B*, written as $\alpha \upharpoonright B$, is the unique assembly satisfying $\text{dom}\,(\alpha \upharpoonright B) = B \cap \text{dom }\alpha$. If π is a sequence over $\mathbb{Z}^2$ (such as a path), then we write $\alpha \upharpoonright \pi$ to mean α restricted to the set of points in π. If $A \subseteq \text{dom }\alpha$, we write $\alpha \setminus A = \alpha \upharpoonright (\text{dom }\alpha - A)$. If $\vec{v} \in \mathbb{Z}^2$, then the *translation of α by $\vec{v}$* is defined as the assembly $(\alpha + \vec{v})$ satisfying, for all $\vec{a} \in \mathbb{Z}^2$,

$$(\alpha + \vec{v})\,(\vec{a}) = \begin{cases} \alpha(\vec{a} - \vec{v}) & \text{if } \vec{a} - \vec{v} \in \text{dom }\alpha \\ \uparrow & \text{otherwise,} \end{cases}$$

where $\uparrow$ signifies that the function is undefined.

A *grid graph* is a graph $G = (V, E)$ in which $V \subseteq \mathbb{Z}^2$ and every edge $\{\vec{a}, \vec{b}\} \in E$ has the property that $\vec{a} - \vec{b} \in U_2$. The *binding graph of* an assembly α is the grid graph $G_\alpha = (V, E)$, where $V = \text{dom }\alpha$, and $\{\vec{m}, \vec{n}\} \in E$ if and only if (1) $\vec{m} - \vec{n} \in U_2$, (2) $\text{col}_{\alpha(\vec{m})}(\vec{n} - \vec{m}) = \text{col}_{\alpha(\vec{n})}(\vec{m} - \vec{n})$, and (3) $\text{str}_{\alpha(\vec{m})}(\vec{n} - \vec{m}) > 0$ and $\text{str}_{\alpha(\vec{m})}(\vec{n} - \vec{m}) = \text{str}_{\alpha(\vec{n})}(\vec{m} - \vec{n})$. An assembly is *$\tau$-stable*, where $\tau \in \mathbb{N}$ (this parameter will ultimately be used as the "temperature" of a tile assembly system), if it cannot be broken up into smaller assemblies without breaking bonds of total strength at least τ (i.e., if every cut of G_α cuts edges, the sum of whose strengths is at least τ).

Self-assembly begins with a *seed assembly* σ (typically assumed to be finite and τ-stable) and proceeds asynchronously and nondeterministically, with tiles adsorbing one at a time to the existing assembly in any manner that preserves stability at all times.

A *tile assembly system* (*TAS*) is an ordered triple $\mathcal{T} = (T, \sigma, \tau)$, where T is a finite set of tile types, σ is a seed assembly with finite domain, and τ is the temperature. In subsequent sections of this paper, we assume that $\tau = 1$ unless explicitly stated otherwise. An *assembly sequence* in a TAS $\mathcal{T} = (T, \sigma, 1)$ is a (possibly infinite) sequence $\vec{\alpha} = (\alpha_i \mid 0 \le i < k)$ of assemblies in which $\alpha_0 = \sigma$ and each α_{i+1} is obtained from α_i by the "τ-stable" addition of a single tile, that is, the new tile binds to α_i (thus forming α_{i+1}) with strength at least τ. The *result* of an assembly sequence $\vec{\alpha}$ is the unique assembly $\alpha = \text{res}(\vec{\alpha})$ satisfying $\text{dom res}(\vec{\alpha}) = \bigcup_{0 \le i < k} \text{dom }\alpha_i$ and, for each $0 \le i < k$, $\alpha_i \sqsubseteq \text{res}(\vec{\alpha})$. Here we say that α is a *producible assembly.*

We write $\mathcal{A}[\mathcal{T}]$ for the *set of all producible assemblies of* $\mathcal{T}$. An assembly α is *terminal*, and we write $\alpha \in \mathcal{A}_\Box[\mathcal{T}]$, if no tile can be stably added to it. We write $\mathcal{A}_\Box[\mathcal{T}]$ for the *set of all terminal assemblies of* $\mathcal{T}$. A TAS $\mathcal{T}$ is *directed* (a.k.a. *deterministic*, *confluent*, *produces a unique assembly*) if it has exactly one terminal assembly i.e., $|\mathcal{A}_\Box[\mathcal{T}]| = 1$. The reader is cautioned that the term "directed" has also been used for a different, more specialized notion in self-assembly [1].

Note that, if $\mathcal{T} = (T, \sigma, 1)$ and $\alpha \in \mathcal{A}[\mathcal{T}]$, then there is a finite path in the binding graph G_α

from the seed to a point $\vec{x} \in \text{dom } \alpha$, denoted as $\pi_{\vec{0},\vec{x}}$ if and only if there is an assembly sequence $\vec{\alpha}$ satisfying $\text{res}(\vec{\alpha}) = \alpha \restriction \pi_{\vec{0},\vec{x}}$.

A set $X \subseteq \mathbb{Z}^2$ *weakly self-assembles* if there exists a TAS $\mathcal{T} = (T, \sigma, 1)$ and a set $B \subseteq T$ (B constitutes the "black" tiles) such that $\alpha^{-1}(B) = X$ holds for every assembly $\alpha \in \mathcal{A}_\Box[\mathcal{T}]$. A set X *strictly self-assembles* if there is a TAS $\mathcal{T}$ for which every assembly $\alpha \in \mathcal{A}_\Box[\mathcal{T}]$ satisfies $\text{dom } \alpha = X$. Note that if X strictly self-assembles, then X weakly self-assembles. (Let all tiles be black.)

3 Pumpability, Finite Closures, and Combs

Throughout this section, let $\mathcal{T} = (T, \sigma, 1)$ be a directed TAS, and α be the unique assembly satisfying $\alpha \in \mathcal{A}_\Box[\mathcal{T}]$. Further, we assume without loss of generality that σ consists of a single "seed" tile type placed at the origin.

Given $\vec{0} \neq \vec{v} \in \mathbb{Z}^2$, a *$\vec{v}$-semi-periodic path in α originating at $\vec{a}_0 \in \text{dom } \alpha$* is an infinite, simple path $\pi = (\vec{a}_0, \vec{a}_1, \ldots)$ in the binding graph G_α such that there is a constant $k \in \mathbb{N}$ such that, for all $j \in \mathbb{N}$, $\pi[j+k] = \pi[j] + \vec{v}$, and $\alpha(\pi[j+k]) = \alpha(\pi[j])$. Intuitively, $\vec{v}$ is the "geometric" period of the path – the straight-line vector between two repeating tile types – while k is the "linear" period – the number of tiles that must be traversed along the path before the tile types repeat, which is at least $||\vec{v}||_1$, but possibly larger if the segment from $\pi[j]$ to $\pi[j] + \vec{v}$ is "winding".

An *eventually $\vec{v}$-semi-periodic path in α originating at $\vec{a}_0 \in \text{dom } \alpha$* is an infinite, simple path $\pi = (\vec{a}_0, \vec{a}_1, \ldots)$ in the binding graph G_α for which there exists $s \in \mathbb{N}$ such that the path $\pi' = (\pi[s], \pi[s+1], \ldots)$ is a v-semi-periodic path in α originating at $\pi[s]$. Let the *initial segment length* be the smallest index s^* such that $\pi'' = (\pi[s^* - k], \pi[s^* - k + 1], \ldots)$, where k retains its meaning from the previous paragraph, is a $\vec{v}$-semi-periodic path originating at the point $\pi[s^* - k]$. The *initial segment of* π is the path $\pi[0 \ldots s^* - 1]$ (for technical reasons, we enforce the initial segment of π to contain the simple path $\pi[0 \ldots s^* - k - 1]$ along with one period of π''). The *tail of* π is $\pi[s^* \ldots]$. Note that the tail of an eventually $\vec{v}$-semi-periodic path is simply a $\vec{v}$-semi-periodic path originating at $\pi[s^*]$. An *eventually $\vec{v}$-semi-periodic path in* α is an eventually $\vec{v}$-semi-periodic path in α originating at $\vec{a}_0$ for some $\vec{a}_0 \in \text{dom } \alpha$. We say that π is a *$\vec{v}$-periodic* path in α if $\pi = (\ldots, \vec{a}_{-1}, \vec{a}_0, \vec{a}_1, \ldots)$ is a two-way infinite simple path such that, for all $j \in \mathbb{Z}$, $\alpha(\pi[j] + \vec{v}) = \alpha(\pi[j])$.

Let $\vec{w}, \vec{x} \in \text{dom } \alpha$, π be a simple path from $\vec{w}$ to $\vec{x}$ in the binding graph G_α, and $i, j \in \mathbb{N}$ with $0 \leq i < j \leq |\pi|$. We say that π has a *pumpable segment* $\pi[i \ldots k]$ (with respect to $\vec{v} = \pi[j] - \pi[i]$) if there exists a $\vec{v}$-semi-periodic path π' in α originating at $\pi[i]$ and $\pi'[0 \ldots j - i] = \pi[i \ldots j]$. Note that the linear period here is $|\pi'[0 \ldots j - i]|$.

Intuitively, the path π has a pumpable segment if, after some initial sequence of tile types, it consists of a subsequence of tile types which is repeated in the same direction an infinite number of times, one after another. Note that our definition of a pumpable segment applies only to directed tile assembly systems, as the definition is based on α, the unique terminal assembly. That is, a segment π is pumpable if in the final assembly α, the infinite path that results from pumping π appears as a subassembly of α. Figure 1 shows an assembly in which the same tile type repeats along a path, but the segment between the occurrences is not pumpable.

Let $c \in \mathbb{N}$ and $\vec{v} \in \mathbb{Z}^2$. The *diamond of radius c centered about the point $\vec{v}$* is the set of points defined as $D(c, \vec{v}) = \{\ (x, y) + \vec{v} \mid |x| + |y| \leq c\ \}$. Let $\vec{w}, \vec{x} \in \text{dom } \alpha$, $c \in \mathbb{N}$, and π be a simple path from $\vec{w}$ to $\vec{x}$ in the binding graph G_α. We say that π is a *pumpable path from $\vec{w}$ to $\vec{x}$ in α* if it contains a pumpable segment $\pi[i \ldots k]$ for some $i, k \in \mathbb{N}$ such that $0 \leq i < k \leq |\pi|$. Given $c \in \mathbb{N}$, we say that $\mathcal{T}$ is *c-pumpable* if, for every $\vec{w}, \vec{x} \in \text{dom } \alpha$ with $\vec{x} \notin D(c, \vec{w})$, there exists a pumpable

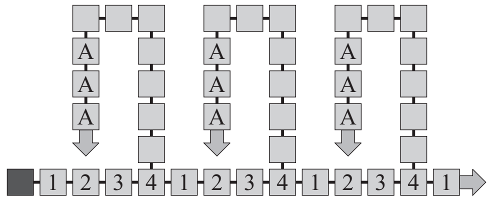


Figure 1: An assembly containing a path with repeating tiles A-A that do *not* form a pumpable segment, because they are blocked from infinite growth by the existing assembly. Note, however, that any sufficiently long path from the origin (at the left) contains a pumpable segment, namely the repeating segment 1-2-3-4-1 along the bottom row, which can be pumped infinitely to the right (and, since the system is directed, *is* pumped infinitely to the right).

path π from $\vec{x}$ to $\vec{w}$ in α. We say that $\mathcal{T}$ is *pumpable* if it is c-pumpable for some $c \in \mathbb{N}$.

Observation 3.1. *If a directed TAS $\mathcal{T} = (T, \sigma, 1)$ has the property that no adjacent tiles in the terminal assembly have mismatched glues, then $\mathcal{T}$ is pumpable.*

Proof. Let α be the unique terminal assembly of $\mathcal{T}$. Let π be any path in G_α of length at least $|T|+1$, and let $\vec{u}, \vec{v}$ be two positions of π with the same tile type. If $\mathcal{T}$ is not pumpable, then attempting to repeat π will cause a "collision" with some other position in α; i.e., eventually the repetition of π will result in attempting to place a tile type t at a position $\vec{x}$ at which $\alpha(\vec{x}) \neq t$. Let p be the tile type at the position immediately preceding $\vec{x}$ on the copy of π at which this occurs. Then the glue of p on the side facing $\vec{x}$ must mismatch with the opposite glue of $\alpha(\vec{x})$, or else in any *previous* repetition of π, t and $\alpha(\vec{x})$ would nondeterministically compete to bind to p, violating the directedness of $\mathcal{T}$. Since we assumed $\mathcal{T}$ is directed and has no mismatches, this implies $\mathcal{T}$ must be c-pumpable with $c = |T|+1$. $\square$

Observation 3.1, together with Theorem 4.3, provides an infinitary analog of the lower bounds of [8, 9].

Figure 2 shows, from left to right, (1) a partially complete assembly beginning from the (dark grey) seed tile, where the dark notches between adjacent tiles represent strength 1 bonds, and a tile selected for the example, (2) the full path leading from the seed to the selected tile, (3) the tile types for a segment of the path, showing the repeating pattern of tile types '1-2-3-4', and (4) an extended version of the path which shows its ability to be pumped.

In our proof, it is helpful to consider extending an assembly in such a way that no individual tile in the existing assembly is extended by more than a finite amount (though an infinite assembly may have an infinite number of tiles that can each be extended by a finite amount). We call such an extension the *finite closure* of the assembly, and define it formally as follows. Let $\alpha' \in \mathcal{A}[\mathcal{T}]$ and let $\alpha \in \mathcal{A}_\Box[\mathcal{T}]$. We say that the *finite closure* of α' is the unique assembly $\mathcal{F}(\alpha')$ satisfying

1. $\alpha' \sqsubseteq \mathcal{F}(\alpha')$, and
2. $\mathrm{dom}\, \mathcal{F}(\alpha')$ is the set of all points $\vec{x} \in \mathbb{Z}^2$ such that every infinite simple path in the binding graph G_α containing $\vec{x}$ intersects $\mathrm{dom}\, \alpha'$.

Intuitively, this means that if we extend α' by only those "portions" that will eventually stop growing, the finite closure is the super-assembly that will be produced. That is, any attempt to "leave" α' through the finite closure and go infinitely far will eventually run into α' again. If α' is

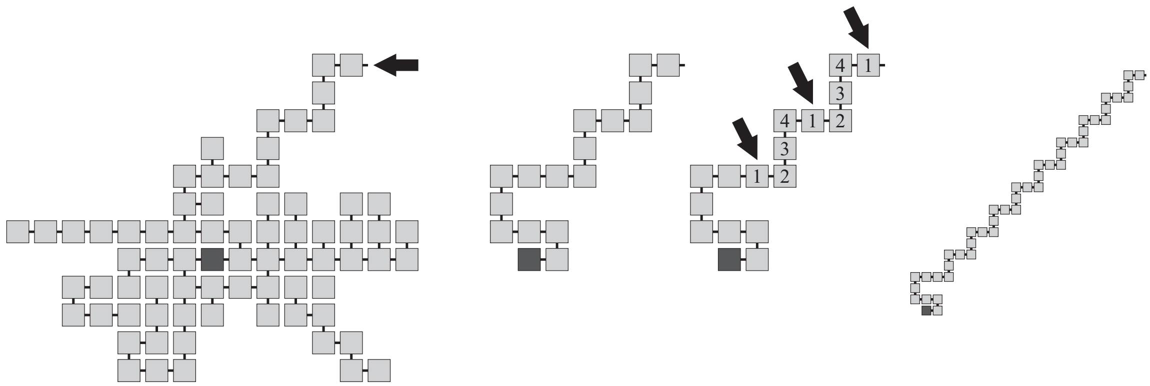
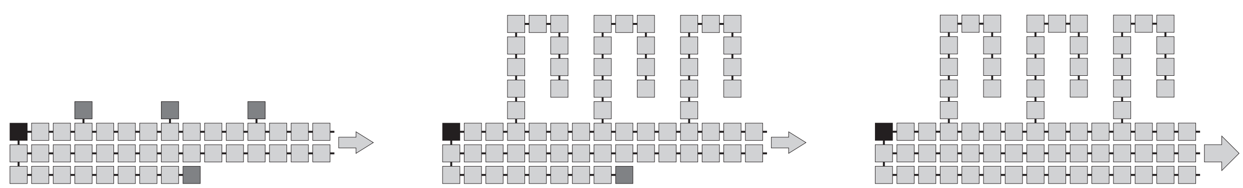


Figure 2: A partial assembly, selected path, pumpable segment, and pumpable path

terminal, then α' is its own finite closure. Note that in general, the finite closure of an assembly α' is not the result of adding finitely many tiles to α'. For instance, if infinitely many points of α' allow exactly one tile to be added, the finite closure adds infinitely many points to α'. However, the finite closure of a finite assembly is always a finite assembly.

Note that $\mathcal{F}(\alpha')$ is always connected (because it contains points that are only reachable from α' via simple paths). It is also the case that there exists an assembly sequence from α' to $\mathcal{F}(\alpha')$ because of $\tau = 1$.



(a) α' (b) $\mathcal{F}(\alpha')$ (c) α

Figure 3: Example of a finite closure. The dark gray points represent locations at which tiles can attach.

For an example of a finite closure of an assembly, see Figure 3. Figure 3c shows the terminal assembly which consists of three rows of tiles that continue infinitely to the right (denoted by the arrow), with a 10 tile upward projection occurring at every fourth column. Figure 3a shows an assembly α' which consists of two rows of tiles continuing infinitely to the right but with an incomplete bottom row and none of the full upward projections. Figure 3b shows $\mathcal{F}(\alpha')$, which is the finite closure of α'. Notice that an infinite number of tiles were added to α' to create $\mathcal{F}(\alpha')$ since an infinite number of upward projections were added, each consisting of only 10 tiles. However, also note that the bottom row is not grown because that row consists of an infinite path and therefore cannot be part of the finite closure.

Let $\pi^{\rightarrow}$ be an eventually $\vec{v}$-semi-periodic path in α originating at $\vec{0}$ where $\vec{0} \neq \vec{v} \in \mathbb{Z}^2$. Let $\vec{u} \in \mathbb{Z}^2$ such that $\vec{u} \neq z \cdot \vec{v}$ for all $z \in \mathbb{R}$. Suppose that there is a point $\vec{b}$ on the tail of $\pi^{\rightarrow}$ such that there is an eventually $\vec{u}$-semi-periodic path $\pi^{\uparrow}$ in α originating at $\vec{b}$ such that $\pi^{\rightarrow} \cap \pi^{\uparrow} = \left\{\vec{b}\right\}$. Define the assembly $\alpha^* = \alpha \restriction \left(\pi^{\rightarrow} \cup \bigcup_{n \in \mathbb{N}} \left(\pi^{\uparrow} + n \cdot \vec{v}\right)\right)$. It is easy to see that α^* need not be a

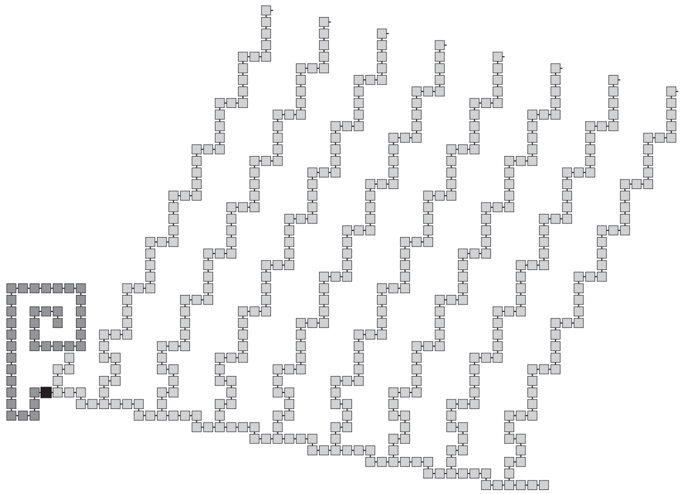


Figure 4: Example of a comb connected to a hard-coded assembly (dark tiles that spiral). The seed is the tile at the center of the spiral. The comb originates at the bottom most black tile. Note that a finite number of "attempts at teeth" (one in the above example) may be blocked before the infinite teeth are allowed to grow.

producible assembly. We say that α^* is a *comb in* (with respect to $\pi^{\rightarrow}$ and $\pi^{\uparrow}$) if, for every $n \in \mathbb{N}$, $\alpha^* \upharpoonright \pi^{\uparrow} + n \cdot \vec{v} = \alpha^* \upharpoonright \left(\pi^{\uparrow} + n \cdot \vec{v}\right)$. We refer to the assembly $\alpha^* \upharpoonright \pi$ as the *base* of α^*. For any $n \in \mathbb{N}$, we define the n^{th} *tooth of* α^* to be the assembly $\alpha \upharpoonright \left(\pi^{\uparrow} + n \cdot \vec{v}\right)$. We say that the comb α^* *starts at the point* $\pi[s]$ (the point at which the eventually-periodic path $\pi^{\rightarrow}$ becomes periodic). It follows from the definition that, if α^* is a comb in α, then $\alpha^* \in \mathcal{A}[\mathcal{T}]$. Note that here we use $\uparrow$ as a "decoration" signifying the direction of a tooth of a comb rather than to mean "undefined."

See Figure 4 for an example of a comb. A comb, intuitively, is a generalization of the assembly in which an infinite periodic one-way path (the base) grows along the positive x-axis, and once per period, an infinite periodic path (a tooth) grows in the positive y direction, creating an infinite number of "teeth". The generalizations are that (1) the base and teeth need not run parallel to either axis, and (2), the teeth may have some initial hard-coded tiles before the repeating periodic segment begins. Also, it is possible at temperature 1 to build multiple combs with the same base, but with different teeth, growing in either direction.

4 Main Result

We show in this section that only "simple" sets weakly self-assemble in directed, pumpable tile assembly systems at temperature 1. We now formally define "simple."

Definition 4.1. A set $X \subseteq \mathbb{Z}^2$ is *linear* if there exist three vectors $\vec{b}$, $\vec{u}$, and $\vec{v}$ such that

$$X = \left\{ \vec{b} + n \cdot \vec{u} + m \cdot \vec{v} \;\middle|\; n, m \in \mathbb{N} \right\}.$$

A set is *semilinear* if it is a finite union of linear sets.

Note that a set that is periodic along only one dimension is also linear, since this corresponds to the condition that exactly one of the vectors $\vec{u}$ or $\vec{v}$ is equal to $\vec{0}$ (or if $\vec{u}$ and $\vec{v}$ are parallel). Similarly, if $\vec{u} = \vec{v} = \vec{0}$, then the definition of linear is equivalent to A being a singleton set.

The following observation (as well as, for example, [5]) justifies the intuition that semilinear sets constitute only the computationally simplest subsets of $\mathbb{Z}^2$.

Observation 4.2. *Let $A \subseteq \mathbb{Z}^2$ be a semilinear set. Then the unary languages*

$$L_{A,x} = \left\{ \ 0^{|x|} \ \middle| \ (x, y) \in A \text{ for some } y \in \mathbb{Z} \ \right\}$$

and

$$L_{A,y} = \left\{ \ 0^{|y|} \ \middle| \ (x, y) \in A \text{ for some } x \in \mathbb{Z} \ \right\}$$

consisting of the unary representations of the projections of A onto the x-axis and y-axis, respectively, are regular languages.

Proof. Let $B \subseteq \mathbb{Z}^2$ be a linear set. It is routine to verify that B's projection along the x-axis is a (singly) periodic set; i.e., a set

$$B_x = \{ \ x \in \mathbb{Z} \mid (x, y) \in B \text{ for some } y \in \mathbb{Z} \ \}$$

such that there is a number $v \in \mathbb{N}$ such that, for all $x \in \mathbb{Z}$, $x \in B_x \implies x+v \in B_x$. It is well-known that a unary language $L \in \{0\}^*$ is regular if and only if the set $N = \{ \ n \in \mathbb{N} \mid 0^n \in L \ \}$ of lengths of strings in L is eventually periodic. Therefore the language $L_{B,x} = \left\{ \ 0^{|n|} \ \middle| \ n \in B_x \ \right\}$ is regular. A symmetric argument establishes that $L_{B,y}$ is a regular language as well, so the theorem holds for any linear set. Since A is a finite union of linear sets, the theorem follows by the closure of the regular languages under finite union. □

The following theorem is the main result of this paper.

Theorem 4.3. *Let $\mathcal{T} = (T, \sigma, 1)$ be a directed, pumpable TAS. If a set $X \subseteq \mathbb{Z}^2$ weakly self-assembles in $\mathcal{T}$, then X is a semilinear set. Conversely, every semilinear set weakly self-assembles at temperature 1.*

The proof idea of the forward direction of Theorem 4.3 is as follows. (The reverse direction is easy.) Suppose that $\alpha \in \mathcal{A}_{\Box}[\mathcal{T}]$. Note that α is unique since $\mathcal{T}$ is directed. Either α is an infinite "grid" that fills the plane, or there exists finitely many linear combs and paths that, taken together, "cover" every point in X (in the sense that each such point is in the finite closure of one of these combs or paths).

The reason for this is that each comb is defined by two vectors $\vec{u}$ (the base) and $\vec{v}$ (the teeth), and these vectors form a "basis" for the space of points located within the cone formed by the base and the first tooth of the comb. While the vectors do not reach every point in this cone, they reach within a constant distance of every point in the cone, and the semilinear regularity of the teeth and base enforces semilinear regularity in between the teeth as well. Of course, not all combs have teeth, in which case the comb is just a periodic path.

We associate each black tile with some periodic path or comb that begins in a fixed radius about the origin (utilizing the fact that a path cannot go far from the origin without having pumpable segments that can be used to construct a periodic path or comb). This association is via finite closure: each black tile is on the finite closure of a periodic path or comb close to the origin. The finite number of combs and periodic paths originating within this radius tells us that the number of semilinear sets of (locations tiled by) black tiles that they each define is finite. To see that there are only a finite number of periodic paths or combs originating in a fixed radius around the origin, observe that there are only a finite number of "slopes" (vectors) possible to construct from any two points in a fixed radius, each periodic path is defined by one such vector, and each comb is defined by two such vectors.

The remainder of this section is devoted to proving Theorem 4.3. We will now prove a series of technical lemmas that will reveal "order" in the seemingly disordered realm of temperature 1 (a.k.a., non-cooperative) self-assembly.

Definition 4.4. Let α_1, α_2 be assemblies. We say that α_1 and α_2 are *consistent* if, for all $\vec{v} \in \text{dom } \alpha_1 \cap \text{dom } \alpha_2$, $\alpha_1(\vec{v}) = \alpha_2(\vec{v})$.

Definition 4.5. Let α_1, α_2 be consistent assemblies. The *union of* α_1 *and* α_2, written as $\alpha_1 \cup \alpha_2$, is the unique assembly $\alpha_1 \cup \alpha_2$ satisfying, $\text{dom}\,(\alpha_1 \cup \alpha_2) = \text{dom } \alpha_1 \cup \text{dom } \alpha_2$ and $\alpha_1 \sqsubseteq \alpha_1 \cup \alpha_2$ and $\alpha_2 \sqsubseteq \alpha_1 \cup \alpha_2$.

Observation 4.6. *Let $\mathcal{T} = (T, \sigma, 1)$ be a directed TAS. If α_1 and α_2 are each connected as well as consistent with each other, and $\alpha_1 \in \mathcal{A}[\mathcal{T}]$, then $\alpha_1 \cup \alpha_2 \in \mathcal{A}[\mathcal{T}]$.*

The next lemma states that, if we wish to show that the existence of a semi-periodic path in the assembly implies a periodic path (i.e., it is not only one-way infinite but two-way infinite, bisecting the plane), it suffices to show that some translation – along the semi-periodic path – of the assembly connecting the seed to the semi-periodic-path, is consistent with the semi-periodic path, namely, the same assembly leading from the seed to the semi-periodic path appears further along the path. See Figure 5 for an example of this situation, and in particular, pay attention to $\pi_{\vec{a}}^{\rightarrow}$ along with the dark tiles connecting the seed to this path. In this case, if the assembly connecting the seed to $\pi_{\vec{a}}^{\rightarrow}$ was also found translated further down the path, then the two would intersect–and if the two assemblies were consistent at their points of intersection, then the next lemma says that $\pi_{\vec{a}}^{\rightarrow}$ is actually a (two-way) periodic path.

Lemma 4.7. *Let $\mathcal{T} = (T, \sigma, 1)$ be a directed TAS, $\vec{0} \neq \vec{v} \in \mathbb{Z}^2$, α be the unique assembly satisfying $\alpha \in \mathcal{A}_{\Box}[\mathcal{T}]$, and $\vec{a} \in \text{dom } \alpha$. Let $\pi_{\vec{a}}^{\rightarrow}$ be a $\vec{v}$-semi-periodic path in α originating at $\vec{a}$, and let $\pi_{\vec{0},\vec{a}}$ be a simple finite path in G_α from $\vec{0}$ to $\vec{a}$. Let $c > |\pi_{\vec{0},\vec{a}}|$ be a positive integer. If $\left(\alpha \restriction \pi_{\vec{0},\vec{a}}\right) + c \cdot \vec{v}$ is consistent with $\alpha \restriction \pi_{\vec{a}}^{\rightarrow}$, then there is a (two-way infinite) $\vec{v}$-periodic path in G_α containing $\vec{a}$.*

Proof. Since $c > \left|\pi_{\vec{0},\vec{a}}\right|$, $\text{dom}\left(\left(\alpha \restriction \pi_{\vec{0},\vec{a}}\right) + c \cdot \vec{v}\right) \cap \pi_{\vec{a}}^{\rightarrow}$ is maximal over all $c \in \mathbb{N}$. Since $\left(\alpha \restriction \pi_{\vec{0},\vec{a}}\right) + c \cdot \vec{v}$ is consistent with $\alpha \restriction \pi_{\vec{a}}^{\rightarrow}$, extending the tiles of $\pi_{\vec{a}}^{\rightarrow}$ in the direction $-\vec{v}$ will intersect $\pi_{\vec{0},\vec{a}}$ only at each position where the existing tile in $\alpha \restriction \pi_{\vec{0},\vec{a}}$ agrees with the extension. Therefore such an extension will not be blocked by $\alpha \restriction \pi_{\vec{0},\vec{a}}$, and $\pi_{\vec{a}}^{\rightarrow}$ is merely one half of a (two-way) periodic path in α. $\Box$

The next lemma states that if two parallel semi-periodic paths are connected through their pumpable segments in "another way besides the trivial way" (since all points are connected by *some* path in the binding graph), then this connection can be exploited to show that the paths must actually form two-way periodic paths that bisect the plane.

Lemma 4.8. *Let $\mathcal{T} = (T, \sigma, 1)$ be a directed TAS, $\vec{0} \neq \vec{v} \in \mathbb{Z}^2$, α be the unique assembly satisfying $\alpha \in \mathcal{A}_{\Box}[\mathcal{T}]$, $\vec{a} \in \text{dom } \alpha$ and $\pi_{\vec{0},\vec{a}}$ be a simple finite path from $\vec{0}$ to $\vec{a}$ in α. If $\pi_{\vec{a}}^{\rightarrow}$ is a $\vec{v}$-semi-periodic* *path in G_α originating at $\vec{a}$, $\vec{d} \in \mathrm{dom}\,\alpha$ such that $\pi_{\vec{d}}^{\rightarrow}$ is a $z \cdot \vec{v}$-semi-periodic path in α originating at $\vec{d}$, for some $z \in \mathbb{R}^+$, and there is a simple path from some point $\vec{c}$ on the tail of $\pi_{\vec{a}}^{\rightarrow}$ to $\vec{d}$ in G_α, denoted as $\pi_{\vec{c},\vec{d}}$, with $\pi_{\vec{c},\vec{d}} \cap \left(\pi_{\vec{0},\vec{a}} \cup \pi_{\vec{a}}^{\rightarrow} \cup \pi_{\vec{d}}^{\rightarrow}\right) = \left\{\vec{c}, \vec{d}\right\}$, then there exists a $\vec{v}$-periodic path in α containing $\vec{a}$.*

Intuitively, we prove Lemma 4.8 by showing that an appropriately translated copy of the simple (finite) path from the seed to the origination point of $\pi_{\vec{a}}^{\rightarrow}$ is consistent with $\pi_{\vec{a}}^{\rightarrow}$. See Figure 5 for a visual depiction of (a simple example of) the hypothesis of Lemma 4.8 and Figure 6 for an illustration of the conclusion of Lemma 4.8.

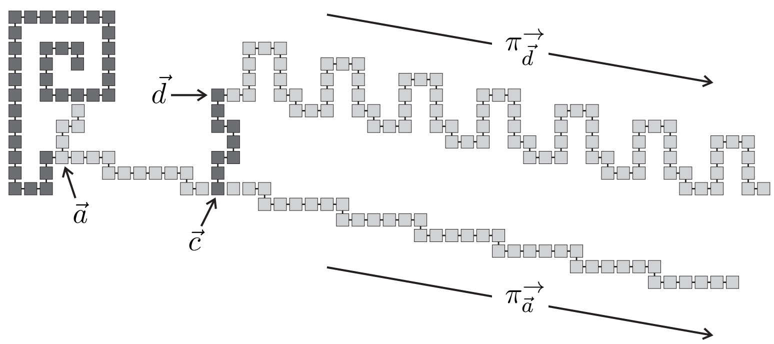


Figure 5: An example of the hypothesis of Lemma 4.8. The seed tile is at the center of the "spiral." The path $\pi_{\vec{c},\vec{d}}$ is represented by the dark tiles that form the "bridge" between the tail of $\pi_{\vec{a}}^{\rightarrow}$ (the lowest $\vec{v}$-periodic path) and some point on $\pi_{\vec{d}}^{\rightarrow}$ (the upper $\vec{v}$-periodic path).

Proof. Let $c \in \mathbb{N}$ such that $c > \left|\pi_{\vec{0},\vec{a}}\right|$. By Lemma 4.7, it suffices to show that $\left(\alpha \restriction \pi_{\vec{0},\vec{a}}\right) + c \cdot \vec{v}$ is consistent with $\alpha \restriction \pi_{\vec{a}}^{\rightarrow}$. If consistency holds, then we are done, so assume otherwise. This means that there is a point

$$\vec{b} \in \left(\pi_{\vec{0},\vec{a}} + c \cdot \vec{v}\right) \cap \pi_{\vec{a}}^{\rightarrow},$$

satisfying the following two conditions.

1. $\left(\left(\alpha \restriction \pi_{\vec{0},\vec{a}}\right) + c \cdot \vec{v}\right)\left(\vec{b}\right) \neq \left(\alpha \restriction \pi_{\vec{a}}^{\rightarrow}\right)\left(\vec{b}\right)$, and
2. $\vec{b}$ is the "closest" point to $\vec{a} + c \cdot \vec{v}$ on the path $\pi_{\vec{0},\vec{a}} + c \cdot \vec{v}$.

Moreover, the assumption that $\pi_{\vec{c},\vec{d}} \cap \left(\pi_{\vec{0},\vec{a}} \cup \pi_{\vec{a}}^{\rightarrow} \cup \pi_{\vec{d}}^{\rightarrow}\right) = \left\{\vec{c}, \vec{d}\right\}$, along with the existence of the $\vec{v}$-periodic path $\pi_{\vec{d}}^{\rightarrow}$, can be used to show that there exists a simple cycle C in G_α that contains the point $\vec{b}$ such that not every simple (finite) path from $\vec{0}$ to (any point in) C goes through $\vec{b}$. This is because, for any $k \in \mathbb{N}$, $\alpha \restriction \left(\pi_{\vec{c},\vec{d}} + k \cdot \vec{v}\right)$ must be consistent with $\alpha \restriction \pi_{\vec{d}}^{\rightarrow}$ since $\mathcal{T}$ is directed. Then we can define an assembly sequence where (the tile placed at) the input neighbor of $\vec{b}$ is the same as the (tile placed at) the output neighbor of $\vec{b} - c \cdot \vec{v}$ in the assembly sequence resulting in $\alpha \restriction \pi_{\vec{0},\vec{a}}$. Since $\mathcal{T}$ is directed, these must be the same tiles, whence there can be no "first" point of inconsistency between $\alpha \restriction \left(\pi_{\vec{0},\vec{a}} + c \cdot \vec{v}\right)$ and $\alpha \restriction \pi_{\vec{a}}^{\rightarrow}$. $\square$

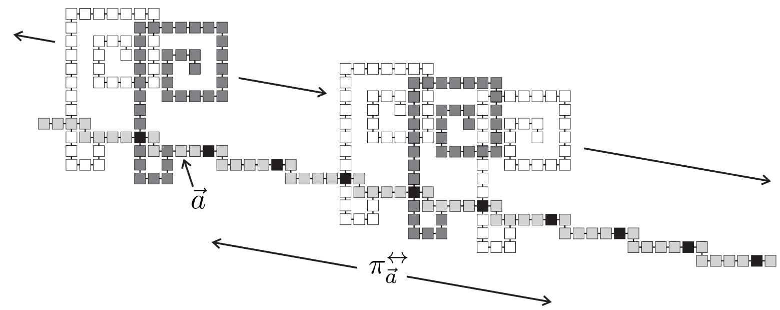


Figure 6: An example of the conclusion of Lemma 4.8. Note that the path $\pi^{\rightarrow}_{\vec{a}}$ is a (two-way infinite) $\vec{v}$-periodic path, which means that (copies of) the path $\pi_{\vec{0},\vec{a}}$ can be translated in the $\pm\vec{v}$ direction.

Definition 4.9. Let α be an assembly. We say α is *doubly periodic* if there exist $\vec{u}, \vec{v} \in \mathbb{Z}^2$ such that $\vec{u} \neq \vec{v}$, $\vec{u} \neq \vec{0}$, $\vec{v} \neq \vec{0}$, and for all $\vec{a} \in \mathbb{Z}^2$, $\alpha(\vec{a}) = \alpha(\vec{a}+\vec{u}) = \alpha(\vec{a}+\vec{v})$, where $\alpha(\vec{a}) = \alpha(\vec{b})$ if both $\alpha(\vec{a})$ and $\alpha(\vec{b})$ are both undefined.

In other words, α is doubly periodic if its values form a repeating "grid" of parallelogram patterns on $\mathbb{Z}^2$, infinite in all directions. It is easy to see that, if α is doubly periodic, then for any set of tile types $B \subseteq T$, dom $\alpha(B)$ (the set that weakly self-assembles) is a union of four linear sets of integer lattice points.

The following technical lemma shows that, if a tail of a tooth of a comb connects to the base through a path other than the "natural" one, then the entire assembly is doubly periodic. This allows us to assume, in the proof of our main theorem, that the teeth of a comb are not interconnected in "inconvenient" ways.

Lemma 4.10. *Let $\mathcal{T} = (T, \sigma, 1)$ be a directed TAS, $\alpha \in \mathcal{A}_{\Box}[\mathcal{T}]$, and α^* be a comb in α. If the tail of the n^{th} tooth is connected to (any point in) the $(n+1)^{st}$ tooth (for $n > 1$) via a simple finite path that does not go through any point on the n^{th} tooth, then α is doubly periodic.*

The proof idea of Lemma 4.10 is as follows. Suppose that α is the unique terminal assembly satisfying $\alpha \in \mathcal{A}_{\Box}[\mathcal{T}]$. Since the tails of two different teeth of the comb α^* are connected via a simple path, we can build an assembly sequence whose result, denoted as $\alpha_{\#}$, is an infinite, plane-filling, doubly periodic assembly. We then show that this assembly is consistent with the tiles near the seed, which means it must be a producible assembly. Finally, since $\mathcal{T}$ is directed and $\alpha_{\#} \sqsubseteq \alpha$, we conclude that α is doubly periodic. Figure 7 illustrates (a simple example of) the hypothesis of the lemma.

Proof. Let α^* be a comb in α with respect to $\pi^{\rightarrow}$ and $\pi^{\uparrow}$, and $\vec{a}$ be the starting point of α^*. Assume that, for non-zero vectors $\vec{u} \neq \vec{v}$, $\pi^{\rightarrow}$ is an eventually $\vec{u}$-semi-periodic path in α originating at $\vec{0}$, and $\pi^{\uparrow}$ is an eventually $\vec{v}$-semi-periodic path in α originating at $\vec{b} \in \pi^{\rightarrow}$ with $\pi^{\rightarrow} \cap \pi^{\uparrow} = \left\{\vec{b}\right\}$.

The hypothesis says that, for some $1 < n \in \mathbb{N}$, there is a simple finite path from some point $\vec{c}$ on the tail of $\pi^{\uparrow} + n \cdot \vec{u}$ to some point $\vec{d} \in \pi^{\uparrow} + (n+1) \cdot \vec{u}$ that does not go through any point in $\pi^{\uparrow} + n \cdot \vec{u}$ except for the first point on $\pi_{\vec{c},\vec{d}}$. Denote this path as $\pi_{\vec{c},\vec{d}}$. Lemma 4.8 tells us that there exists a (two-way infinite) $\vec{v}$-periodic path in α containing the point $\widehat{\vec{b}} = \vec{b} + n \cdot \vec{u}$. Denote this path as $\pi^{\updownarrow}_{\widehat{\vec{b}}}$.

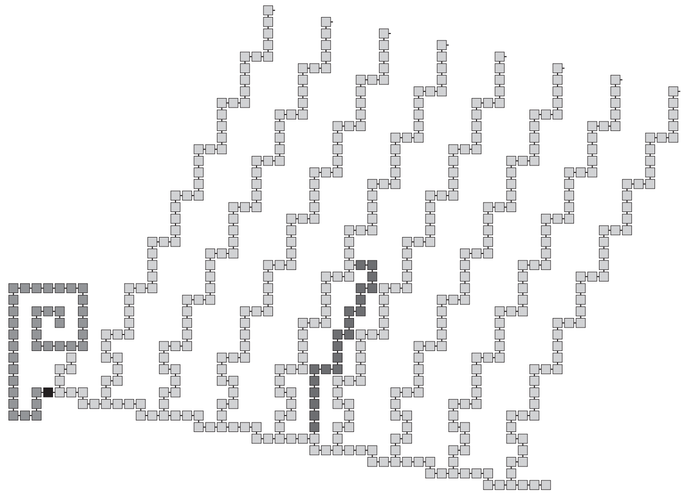


Figure 7: An example of a comb having two teeth connected via a simple path that does not go through the finite closure of the base. The seed tile is the tile in the center of the "spiral".

Let $\pi^{\rightarrow}_{\widehat{\vec{b}}}$ be the $\vec{u}$-semi-periodic path in G_α originating at $\widehat{\vec{b}}$. Since $\pi^{\updownarrow}_{\widehat{\vec{b}}}$ is $\vec{v}$-periodic and $\pi^{\updownarrow}_{\widehat{\vec{b}}} \cap \pi^{\rightarrow} \neq \varnothing$, it must be the case that there is a $\vec{u}$-semi-periodic path, denoted as $\pi^{\rightarrow}_{\widehat{\vec{b}}+\vec{v}}$, in α originating at the point $\widehat{\vec{b}}+\vec{v}$. Let $\pi_{\vec{0},\widehat{\vec{b}}}$ be a finite simple path from $\vec{0}$ to $\widehat{\vec{b}}$ in G_α. If we (re)define $\vec{c} = \widehat{\vec{b}}$, $\vec{d} = \widehat{\vec{b}}+\vec{v}$, and note that there is a simple (finite) path from $\vec{c}$ on the tail of $\pi^{\rightarrow}$ (because $n > 1$) to $\vec{d} \in \pi^{\rightarrow}_{\widehat{\vec{b}}+\vec{v}}$, denoted as $\pi_{\vec{c},\vec{d}}$, in G_α with $\pi_{\vec{c},\vec{d}} \cap \left(\pi_{\vec{0},\widehat{\vec{b}}} \cup \pi^{\rightarrow}_{\widehat{\vec{b}}} \cup \pi^{\rightarrow}_{\widehat{\vec{b}}+\vec{v}}\right) = \left\{\vec{c},\vec{d}\right\}$, then we can again use Lemma 4.8 to conclude that there is a (two-way infinite) $\vec{v}$-periodic path containing $\widehat{\vec{b}}$, hence also containing $\vec{a}$ since those points differ by a multiple of $\vec{v}$. Denote this path as $\pi^{\leftrightarrow}_{\vec{a}}$. If $\pi_{\vec{0},\vec{a}}$ is the initial segment of $\pi^{\rightarrow}$, then the assembly $\alpha \upharpoonright \left(\pi_{\vec{0},\vec{a}} \cup \pi^{\leftrightarrow}_{\vec{a}} \cup \pi^{\updownarrow}_{\vec{b}}\right) \in \mathcal{A}[\mathcal{T}]$.

Define the assembly $\alpha_{\#}$ as follows.

$$\operatorname{dom} \alpha_{\#} = \bigcup_{n\in\mathbb{Z}} ((\alpha \upharpoonright \pi^{\leftrightarrow}_{\vec{a}}) + n\cdot\vec{v}) \cup \bigcup_{n\in\mathbb{Z}} \left(\left(\alpha \upharpoonright \pi^{\updownarrow}_{\vec{b}}\right) + n\cdot\vec{u}\right).$$

Our goal is to show that $\alpha_{\#} \cup \left(\alpha \upharpoonright \pi_{\vec{0},\vec{a}}\right) \in \mathcal{A}[\mathcal{T}]$, which will nearly complete the proof since $\alpha_{\#}$ is doubly periodic.

Let $\alpha_{\text{almost-}\#}$ be the largest assembly satisfying $\alpha_{\text{almost-}\#} \sqsubseteq \alpha_{\#}$ such that $\operatorname{dom} \alpha_{\text{almost-}\#} \cap \pi_{\vec{0},\vec{a}} = \varnothing$ and $G_{\alpha_{\text{almost-}\#}}$ is connected. Note that the assembly $\alpha_{\text{almost-}\#} \cup \left(\alpha \upharpoonright \pi_{\vec{0},\vec{a}}\right) \in \mathcal{A}[\mathcal{T}]$. See Figure 9 for an example of how the assembly $\alpha_{\text{almost-}\#}$ is constructed.

In order to prove that $\alpha_{\#} \cup \left(\alpha \upharpoonright \pi_{\vec{0},\vec{a}}\right) \in \mathcal{A}[\mathcal{T}]$, it suffices to show that $\alpha \upharpoonright \pi_{\vec{0},\vec{a}}$ and $\alpha_{\#}$ are consistent, since $\alpha \upharpoonright \pi_{\vec{0},\vec{a}} \in \mathcal{A}[\mathcal{T}]$ and $\pi_{\vec{0},\vec{a}} \cap \operatorname{dom} \alpha_{\#} \neq \varnothing$. Note that, by the way we constructed $\alpha_{\#}$, not *every* point in $\pi_{\vec{0},\vec{a}}$ can be a point of inconsistency between $\alpha_{\#}$ and $\alpha \upharpoonright \pi_{\vec{0},\vec{a}}$. Fix $n^* \in \mathbb{N}$ such that $\pi_{\vec{0},\vec{a}} \cap \left(\pi_{\vec{0},\vec{a}} + (n^*\cdot\vec{u} + n^*\cdot\vec{v})\right) = \varnothing$, which exists by the finiteness of $\pi_{\vec{0},\vec{a}}$. Since there

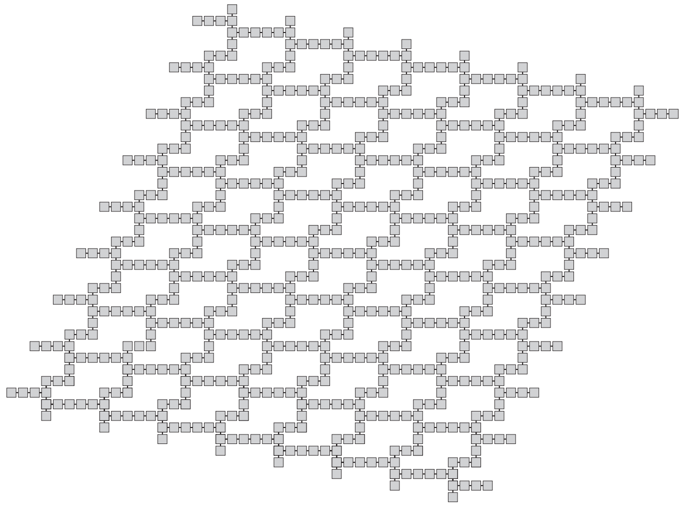
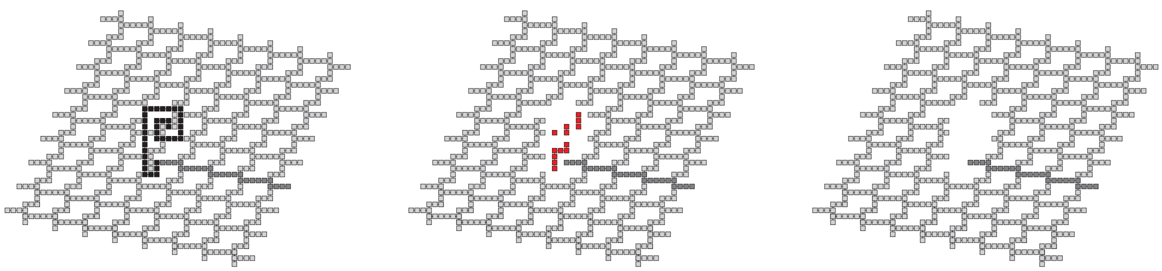


Figure 8: A finite sub-assembly of $\alpha_{\#}$.



(a) $\alpha_{\#} \cup (\alpha \upharpoonright \pi_{\vec{0},\vec{a}})$ (b) $\alpha_{\#} - (\alpha \upharpoonright \pi_{\vec{0},\vec{a}})$ (c) $\alpha_{\mathrm{almost}-\#}$

Figure 9: The black tiles are the path $\pi_{\vec{0},\vec{a}}$. Since $\alpha_{\#} - \left(\alpha \upharpoonright \pi_{\vec{0},\vec{a}}\right)$ may have a finite number of disconnected "islands", we remove these as well to get the largest *connected* subassembly of $\alpha_{\#}$.

exists at least one point $\vec{s} \in \pi_{\vec{0},\vec{a}} \cap \mathrm{dom}\ \alpha_{\#}$ such that $\alpha_{\#}(\vec{s}) = \left(\alpha \upharpoonright \pi_{\vec{0},\vec{a}}\right)(\vec{s})$, we can conclude that

$$\left(\alpha_{\text{almost-}\#} \cup \left(\alpha \upharpoonright \pi_{\vec{0},\vec{a}}\right)\right) \cup \left(\left(\alpha \upharpoonright \pi_{\vec{0},\vec{a}}\right) + n^* \cdot \vec{u} + n^* \cdot \vec{v}\right) \in \mathcal{A}[\mathcal{T}]$$

because $\mathcal{T}$ is directed and every point in $\pi_{\vec{0},\vec{a}} + n^* \cdot \vec{u} + n^* \cdot \vec{v} \cap \mathrm{dom}\ \alpha_{\text{almost-}\#}$ is reachable from $\vec{0}$ via at least two simple finite paths. Thus, $\alpha_{\#}$ and $\alpha \upharpoonright \pi_{\vec{0},\vec{a}}$ must be consistent. Since $\alpha_{\#}$ is trivially doubly periodic, it follows that $\alpha_{\#} \cup \left(\alpha \upharpoonright \pi_{\vec{0},\vec{a}}\right) \in \mathcal{A}[\mathcal{T}]$.

Since $\alpha_{\#} \sqsubseteq \alpha$, it suffices to show that the tiles added to $\alpha_{\#}$ to create α are doubly periodic. Since $\alpha_{\#}$ forms an infinite grid that partitions $\mathbb{Z}^2$ into infinitely many identical "parallelograms", bordered by the same tiles $\Big($those forming a single period of $\pi^{\leftrightarrow}_{\vec{a}}$ and $\pi^{\updownarrow}_{\vec{b}}$, respectively$\Big)$, then by the directedness of $\mathcal{T}$, each of these parallelograms must have the same tiles in the same positions relative to the borders of the parallelogram. Since the parallelogram borders are doubly periodic, the contents within the borders are doubly periodic as well, whence α is doubly periodic. $\square$

The following lemma states that if there is a path from an assembly α' to a point not on the finite closure of α', then there is an eventually periodic path, originating from the same point as the first path, that intersects α' only at the first point.

Lemma 4.11. *Let $\mathcal{T} = (T, \sigma, 1)$ be a pumpable directed TAS, α be the unique assembly satisfying $\alpha \in \mathcal{A}_\Box[\mathcal{T}]$, $\alpha' \sqsubseteq \alpha$, and $\vec{x} \in \text{dom } \alpha$. If $\vec{x} \notin \mathcal{F}(\alpha')$ and there is a finite simple path in G_α that goes through some point $\vec{s} \in \alpha'$ to $\vec{x}$, then, for some $\vec{0} \neq \vec{v} \in \mathbb{Z}^2$, there exists an eventually $\vec{v}$-semi-periodic path π in G_α, originating at $\vec{s}$, with $\pi \cap \text{dom } \alpha' = \{\vec{s}\}$.*

Proof. Let $\vec{r} \in \text{dom } \alpha'$ be a point connected to $\vec{x}$ by a simple finite path $\pi_{\vec{r},\vec{x}}$ satisfying $\pi_{\vec{r},\vec{x}} \cap \text{dom } \alpha' = \{\vec{r}\}$. Let $\vec{p}_0$ be the point on $\pi_{\vec{r},\vec{x}}$ closest to $\vec{x}$ such that G_α has an infinite simple path $\pi_{\vec{p}_0,\infty}$ starting at $\vec{p}_0$ that does not contain the point immediately before $\vec{p}_0$ on $\pi_{\vec{r},\vec{x}}$. Let $\vec{p}_1 \in \text{dom } \alpha \cap \left(\pi_{\vec{p}_0,\infty} - D\left(c, \vec{p}_0\right)\right)$, and $\pi_{\vec{r},\vec{p}_0,\vec{p}_1}$ be a simple path in G_α from $\vec{r}$ to $\vec{p}_1$ that goes through $\vec{p}_0$. Since $\mathcal{T}$ is pumpable, and $\left(\pi_{\vec{p}_0,\infty} - D\left(c, \vec{p}_0\right)\right)$, $\pi_{\vec{r},\vec{p}_0,\vec{p}_1}$ is a pumpable path. Let $\pi_{\vec{r},\vec{p}_0,\vec{p}_1}[i \ldots k]$, for $0 \leq i < k \leq \left|\pi_{\vec{r},\vec{p}_0,\vec{p}_1}\right|$, be the first pumpable segment of $\pi_{\vec{r},\vec{p}_0,\vec{p}_1}$, and $\vec{v} = \pi_{\vec{r},\vec{p}_0,\vec{p}_1}[k] - \pi_{\vec{r},\vec{p}_0,\vec{p}_1}[i]$. The lemma follows by letting $\vec{s} = \pi_{\vec{r},\vec{p}_0,\vec{p}_1}[i]$, and π be the $\vec{v}$-semi-periodic path originating at the point $\vec{s}$ defined by the pumpable segment. $\square$

The next lemma states that if two teeth in a comb are connected in an "inconvenient" way (i.e., other than the trivial ways in which any two points in a stable assembly must be connected), then the entire assembly is an (two-way, two-dimensional) doubly periodic grid.

Lemma 4.12. *Let $\mathcal{T} = (T, \sigma, 1)$ be a pumpable directed TAS, α be the unique assembly satisfying $\alpha \in \mathcal{A}_\Box[\mathcal{T}]$, and α^* be a comb in α. If there exists $\vec{x} \in \text{dom } \alpha$ such that there is a simple finite path from $\vec{0}$ to $\vec{x}$ that goes through the tail of some tooth of α^*, then $\vec{x} \in \mathcal{F}(\alpha^*)$ or α is doubly periodic.*

Proof. Let $\pi^\uparrow$ be the first tooth of α^*. Assume that $\vec{x} \notin \mathcal{F}(\alpha^*)$. It suffices to show that α is doubly periodic. Let $\vec{r}$ be an element of the tail of $n \cdot \vec{u} + \pi^\uparrow$ for some $n \in \mathbb{N}$, such that there exists a simple path in G_α from $\vec{r}$ to $\vec{x}$, denoted as $\pi_{\vec{r},\vec{x}}$, with $\pi_{\vec{r},\vec{x}} \cap \text{dom } \alpha^* = \{\vec{r}\}$. Such a point $\vec{r}$ exists because $\vec{x} \notin \mathcal{F}(\alpha^*)$. By Lemma 4.11, there exists an eventually $\vec{w}$-semi-periodic π in G_α, originating at $\vec{r}$ with $\pi \cap \text{dom } \alpha^* = \varnothing$.

If $\vec{w} \neq z \cdot \vec{v}$ for some $z \in \mathbb{R}^+$, then the $\vec{w}$-semi-periodic path it defines intersects a tooth of α^* and, by Lemma 4.10, α is doubly-periodic. Therefore assume that $\vec{w} = z \cdot \vec{v}$ for some $z \in \mathbb{R}^+$. Let $\vec{a} = \pi[s]$ - the point at which π becomes periodic. Lemma 4.8 tells us that there exists a (two-way infinite) $\vec{w}$-periodic path in α containing the point $\vec{a}$. Denote this path as $\pi_{\vec{a}}^\updownarrow$. Let $\vec{b}$ be the closest point to $\vec{a}$ on the path $\pi_{\vec{a}}^\updownarrow$ that intersects the base of α^*. Let $\widehat{\pi}^\uparrow$ be the $\vec{w}$-semi-periodic path originating at $\vec{b}$ that does not intersect any teeth of α^*. Then we can form a new comb $\widehat{\alpha^*}$ with respect to $\pi^\rightarrow$ and $\widehat{\pi}^\uparrow$. Finally, we can apply Lemma 4.10 to either α^* or $\widehat{\alpha^*}$ in order to conclude that α is doubly periodic. $\square$

The following observation is helpful in the proof of our main theorem, and states the obvious fact that combs define linear sets.

Observation 4.13. *Let $\mathcal{T} = (T, \sigma, 1)$ be a directed pumpable TAS in which the set $X \subseteq \mathbb{Z}^2$ weakly self-assembles, and let α be the unique terminal assembly satisfying $\alpha \in \mathcal{A}_\Box[\mathcal{T}]$. If $\alpha^* \in \mathcal{A}[\mathcal{T}]$ is a comb in α, then the set $X \cap \text{dom } \alpha^*$ is a semilinear set.*

Proof. (*of Theorem 4.3*) Assume the hypothesis. Let $c \in \mathbb{N}$ testify to the c-pumpability of $\mathcal{T}$. We denote by D_1, D_2, and D_3 the diamonds of radius c, $2c$, and $3c$, respectively, around the origin.

We show that each $\vec{x} \in X$ outside of D_2 is part of the finite closure of some periodic path or comb originating in D_3. Therefore, the union of the black tiles of the finite closure of each comb starting in D_3, each periodic path starting in D_3, and each singleton set containing a black tile in D_3, is the union proving the theorem. There are only a finite number of points in D_3, though each could be the origination point of multiple combs or periodic paths. However, if any such point has an infinite number of combs or periodic paths originating from it, then some will intersect at a different point, creating a cycle between the tails of two teeth of combs, and Lemma 4.10 tells us that X is doubly periodic. Otherwise, only a finite number of combs and periodic paths can originate in D_3, each one defining a term in the union, showing that the union is finite. Recall that a singleton set and a semi-periodic path are linear sets, and Observation 4.13 allows us to conclude that the remaining terms representing combs are linear as well.

Let $\vec{x} \in X - D_1$, and α be the unique assembly satisfying $\alpha \in \mathcal{A}_{\Box}[\mathcal{T}]$. Since $\vec{x} \notin D_1$, and because $\mathcal{T}$ is c-pumpable, there exists a c-pumpable path from $\vec{0}$ to $\vec{x}$ in α. Denote this path as $\pi_{\vec{0},\vec{x}}$. Let $\pi_{\vec{0},\vec{x}}[i \ldots k]$, for $0 \leq i < k \leq \left|\pi_{\vec{0},\vec{x}}\right|$, be the first pumpable segment of $\pi_{\vec{0},\vec{x}}$, and $\vec{u} = \pi_{\vec{0},\vec{x}}[k] - \pi_{\vec{0},\vec{x}}[i]$. Let $\pi^{\rightarrow}_{\pi_{\vec{0},\vec{x}}[i]}$ be the $\vec{u}$-semi-periodic path in G_α originating at $\pi_{\vec{0},\vec{x}}[i]$, and $\pi^{\rightarrow} = \pi_{\vec{0},\vec{x}}[0 \ldots i] \cup \pi^{\rightarrow}_{\pi_{\vec{0},\vec{x}}[i]}$.

If $\vec{x} \in \mathrm{dom}\ \mathcal{F}\left(\alpha \upharpoonright \pi^{\rightarrow}\right)$ then we are done because $\mathcal{F}\left(\alpha \upharpoonright \pi^{\rightarrow}\right)$ contains a sub-assembly whose domain is the semi-periodic path $\pi^{\rightarrow}_{\pi_{\vec{0},\vec{x}}[i]}$, which starts in $D_1 \subset D_2$, so assume that $\vec{x} \notin \mathrm{dom}\ \mathcal{F}\left(\alpha \upharpoonright \pi^{\rightarrow}\right)$. The path $\pi^{\rightarrow}$ is the base of the comb we will now construct.

Since $\vec{x} \notin \mathrm{dom}\ \mathcal{F}\left(\alpha \upharpoonright \pi^{\rightarrow}\right)$ and because there is a simple finite path in G_α from some point $\vec{s}$ on the tail of $\pi^{\rightarrow}$ to $\vec{x}$, Lemma 4.11 tells us that there is a $\vec{v}$-semi-periodic path π in G_α originating at $\vec{s}$ with $\pi \cap \pi^{\rightarrow} = \varnothing$. Assume that $\vec{u} \neq z \cdot \vec{v}$ for any $z \in \mathbb{R}$. In this case, let $\pi^{\uparrow} = \pi$. The assembly $\alpha \upharpoonright \pi^{\uparrow}$ is a tooth of some comb with base $\alpha \upharpoonright \pi^{\rightarrow}$. Now define the following assembly.

$$\alpha^* = \alpha \upharpoonright \left(\pi^{\rightarrow} \cup \bigcup_{n \in \mathbb{N}} \left(n \cdot \vec{u} + \pi^{\uparrow} \right) \right).$$

It is clear from the definition that α^* is a comb in α (starting at some point in D_2) with respect to $\pi^{\rightarrow}$ and $\pi^{\uparrow}$. By Lemma 4.12, it follows that $\vec{x} \in \mathcal{F}\left(\alpha^*\right)$ or α is doubly periodic.

We have shown that, assuming $\vec{u} \neq z \cdot \vec{v}$ for any $z \in \mathbb{R}$, every point $\vec{x} \in X - D_1$ is contained in the finite closure of some comb or periodic path originating at a point in D_2 (unless α is doubly periodic). Furthermore, since D_2 is finite, there are at most a finite number of combs. The theorem follows by Observation 4.13.

Recall that π is an eventually $\vec{v}$-semi-periodic path in G_α, and we earlier assumed that $\vec{u} \neq z \cdot \vec{v}$ for any $z \in \mathbb{R}$. Now assume that $\vec{u} = z \cdot \vec{v}$ for some $z \in \mathbb{R}$. Suppose $z < 0$. In this case, let $\pi^{\leftarrow} = \pi$. If $\vec{x} \in \mathcal{F}\left(\alpha \upharpoonright \left(\pi^{\rightarrow} \cup \pi^{\leftarrow}\right)\right)$, then we are done since $\pi^{\leftarrow}$ originates within D_2. If not, then it must be the case that there is a point $\vec{s'}$ on the tail of $\pi^{\leftarrow}$ such that there is a simple finite path from $\vec{s'}$ to $\vec{x}$ in G_α. Then Lemma 4.11 tells us that there is an eventually $\vec{w}$-semi-periodic path π' in G_α originating at $\vec{s'}$ with $\pi' \cap \left(\pi^{\leftarrow} \cup \pi^{\rightarrow}\right) = \varnothing$. Let $\pi^{\nearrow} = \pi'$.

We now have three eventually periodic paths in G_α, $\pi^{\leftarrow}$, $\pi^{\rightarrow}$, and $\pi^{\nearrow}$, whose respective tails are all disjoint. Moreover, at least two of these paths must either be "parallel," or two of them are neither parallel nor anti-parallel, and the argument assuming $\vec{u} \neq z \cdot \vec{v}$ for any $z \in \mathbb{R}$ applies. If they are parallel, then by Lemma 4.8, the base $\pi^{\rightarrow}$ forms a periodic path that cuts the plane into two half-planes. Using reasoning that is similar to the above arguments, it is easy to verify that this implies that α is either

1. the finite closure of a (two-way infinite) periodic path bisecting the plane (hence a finite union of semi-periodic sets),
2. (1) unioned with the finite closure of a comb (with a two-way infinite base) covering one of the half planes formed by the periodic bisection, or
3. (1) unioned with two-way infinite combs on both sides of the bisection, each covering one of the half planes.

In each case, it is clear that the set formed is a semilinear set.

To show the reverse direction, that every semilinear set weakly self-assembles in some temperature 1 tile assembly system (in fact, in a directed, pumpable TAS), let $X = \bigcup_{i=1}^{k} X_i$ be semilinear, with each X_i a linear set. If the "cones" formed by each X_i do not overlap, it is easy to create a TAS that grows hard-coded paths from the seed to each base point of the cone, each of which assembles a comb that weakly assembles the linear set. Consider then if X_i and X_j do have their cones overlap (note that this may not necessarily mean the sets themselves intersect, only that they "interleave").

Then either one cone is contained in the other, or they intersect but each have a region not contained in the other. In either case, the union of the two cones can be partitioned into at most three cones, each of which can be weakly assembled. In the first case, supposing X_i's cone is contained in X_j's cone, the three regions are the portion of X_j to the clockwise direction of X_i, the portion where they intersect, and the portion of X_j to the counterclockwise direction of X_i. The middle portion will need two kinds of black tiles, one for X_i and one for X_j, but in this portion, the set will still be semilinear with periods equal to the least common multiple of the original periods. In the second case, that X_i and X_j have their cones intersect but neither is contained in the other, the three regions are "only X_i", "X_i and X_j", and "only X_j". In either case we have partitioned the union of two cones into three non-overlapping cones, each of which can be weakly assembled by three distinct combs.

This argument is easily extended to the case that more than two linear sets overlap at once. Since only a finite number of linear sets compose X, we have shown that X can be decomposed into a finite number of non-overlapping sets, each of which can be weakly assembled, and each of which is reachable from the origin without intersecting the others. It follows that X weakly self-assembles at temperature 1. □

5 An Application to Discrete Self-Similar Fractals

In this section, we use Theorem 4.3 to show that no discrete self-similar fractal weakly self-assembles in any temperature 1 tile assembly system that is pumpable and directed. Since Winfree [19] showed that one particular discrete self-similar fractal, the discrete Sierpinkski triangle, self-assembles at temperature 2, this provides a concrete example of computation that is possible (and simple) at temperature 2, but impossible at temperature 1, assuming directedness and pumpability.

Definition 5.1. Let $1 < c \in \mathbb{N}$, and $X \subsetneq \mathbb{N}^2$ (we do not consider $\mathbb{N}^2$ to be a self-similar fractal). We say that X is a *c-discrete self-similar fractal*, if there is a non-emptyset set V satisfying $\{(0,0)\} \subset V \subseteq \{0,\dots,c-1\} \times \{0,\dots,c-1\}$ with

$$V \notin \{\{(i,i) \mid 0 \leq i < c\}, \{(i,0) \mid 0 \leq i < c\}, \{(0,i) \mid 0 \leq i < c\}\}$$

such that

$$X = \bigcup_{i=0}^{\infty} X_i,$$

where X_i is the i^{th} *stage* satisfying $X_0 = \{(0,0)\}$, and $X_{i+1} = X_i + c^i V$. In this case, we say that V *generates* X.

Definition 5.2. $X \subsetneq \mathbb{N}^2$. We say that X is a *discrete self-similar fractal* if it is a c-discrete self-similar fractal for some $c \in \mathbb{N}$.

The following observation is clear by Definition 5.1.

Observation 5.3. *If $X \subsetneq \mathbb{N}^2$ is a discrete self-similar fractal then X is not a semilinear set.*

In the following proof of Observation 5.3, we will use a commonly-used dimension for discrete fractals known as *zeta-dimension* (see [4] for a more complete treatment). The *zeta-dimension* of a set $A \subseteq \mathbb{Z}^2$ is

$$\mathrm{Dim}_\zeta(A) = \limsup_{n\to\infty} \frac{\log |A_{\leq n}|}{\log n},$$

where $A_{\leq n} = \{(k,l) \in A \mid |k| + |l| \leq n\}$.

Proof. Let $A = \bigcup_{i=1}^{k} A_i$ be an arbitrary semilinear set, where each A_i is a linear set. If any of the A_i's are defined as $A_i = \left\{ \vec{b} + n\vec{u} + m\vec{v} \;\middle|\; n,m \in \mathbb{N} \right\}$ where $\vec{u}$ and $\vec{v}$ are linearly independent (i.e., A_i has a "comb" structure), then A_i has positive density. Therefore A also has positive density, whence $\mathrm{Dim}_\zeta(A) = 2$. Otherwise, all A_i's are dimension 1 (periodic paths) or 0 (singleton points), whence $\mathrm{Dim}_\zeta(A) \leq 1$. Since X is a discrete self-similar fractal, then $1 < \mathrm{Dim}_\zeta(X) < 2$. Therefore $X \neq A$. $\square$

Theorem 5.4. *Let $X \subsetneq \mathbb{N}^2$ be a discrete self-similar fractal.*

1. *X does not weakly self-assemble in any tile assembly system that is pumpable and directed.*

2. *X does not strictly self-assemble in any tile assembly system that is pumpable and directed.*

Proof. (1) follows directly from Observation 5.3. To prove (2), note that if a set $X \subseteq \mathbb{Z}^2$ weakly self-assembles then, by definition, X strictly self-assembles. $\square$

6 Conclusion

We have studied the class of shapes that self-assemble in Winfree's abstract tile assembly model at temperature 1. We introduced the notion of a pumpable temperature 1 tile assembly system and then proved that, if X weakly self-assembles in a pumpable, directed tile assembly system, then X is necessarily "simple" in the sense that X is merely a semilinear set, a finite union of simple, "two-dimensionally periodic" sets. Finally, we conjecture that our results hold in the absence of the pumpability hypothesis.

Conjecture 6.1. *Let $\mathcal{T} = (T, \sigma, 1)$ be a directed tile assembly system and $\alpha \in \mathcal{A}_\Box[\mathcal{T}]$. If $\mathrm{dom}\ \alpha$ is infinite, then $\mathcal{T}$ is pumpable.*

It is always possible to produce long paths in which the presence of a segment with two repetitions of a tile type does not imply that the segment is pumpable. However, in every case we consider, there is always a previous segment of the path that is pumpable. Proving Conjecture 6.1 would imply that every directed tile assembly system weakly self-assembles a semilinear set.

We also leave open the question of whether the hypothesis of directedness may be removed. We use the property of directedness at many points in our proof, but in some cases, a more careful and technically convoluted argument could be used to show that the tile set need not be directed. Intuitively, an undirected tile set $\mathcal{T}$ that weakly self-assembles a set $X \subseteq \mathbb{Z}^2$ is deterministic in that all terminal assemblies of $\mathcal{T}$ "paint" exactly the points in X black, but is nondeterministic in the sense that different terminal assemblies of $\mathcal{T}$ may place different tiles in the same location (including different black tiles at locations in X), and may even place non-black tiles at locations in one terminal assembly that are left empty in other terminal assemblies. Undirected tile assembly systems that weakly self-assemble a unique set X exist, but in every case that we know of, the undirected tile set may be replaced by a directed tile set self-assembling the same set.

If both hypotheses of directedness and pumpability could be removed from the entire proof, then our main result would settle the case of computation via self-assembly at temperature 1, by showing that every temperature 1 tile assembly system weakly self-assembles a semilinear set if it weakly self-assembles any set at all. As indicated in the introduction, we would interpret this statement, if true, to imply that general-purpose deterministic computation is not possible with two-dimensional temperature 1 tile assembly systems. We emphasize that the notion of weak self-assembly retains a sort of "determinism" in the sense that the locations receiving a black tile are fixed, no matter how assembly proceeds. This prevents the 2D results of Cook, Fu, and Schweller [2] and Adleman, discussed in the introduction, from being a counterexample, as those constructions do not weakly self-assembly any set.

Acknowledgment

We wish to thank Maria Axenovich, Matt Cook, and Jack Lutz for useful discussions. We would also like to thank Niall Murphy, Turlough Neary, Damien Woods, and Anthony Seda for inviting us to present a preliminary version of this research at the International Workshop on The Complexity of Simple Programs, University College Cork, Ireland on December 6th and 7th, 2008. Finally, we want to thank the anonymous reviewers who poured great time and effort into their reviews and helped to make this a much better paper.

References

[1] Leonard M. Adleman, Jarkko Kari, Lila Kari, Dustin Reishus, and Petr Sosík, *The undecidability of the infinite ribbon problem: Implications for computing by self-assembly*, SIAM Journal on Computing **38** (2009), no. 6, 2356–2381.

[2] Matthew Cook, Yunhui Fu, and Robert T. Schweller, *Temperature 1 self-assembly: Deterministic assembly in 3D and probabilistic assembly in 2D*, Tech. Report 0912.0027, Computing Research Repository, 2009.

[3] David Doty, *Randomized self-assembly for exact shapes*, SICOMP: SIAM Journal on Computing, to appear. preliminary version appeared in FOCS 2009.

[4] David Doty, Xiaoyang Gu, Jack H. Lutz, Elvira Mayordomo, and Philippe Moser, *Zeta-Dimension*, Proceedings of the Thirtieth International Symposium on Mathematical Foundations of Computer Science, Springer-Verlag, 2005, pp. 283–294.

[5] Oscar H. Ibarra, *Reversal-bounded multicounter machines and their decision problems*, Journal of the ACM **25** (1978), no. 1, 116–133.

[6] Ming-Yang Kao and Robert T. Schweller, *Randomized self-assembly for approximate shapes*, ICALP 2008: International Colloqium on Automata, Languages, and Programming (Luca Aceto, Ivan Damgård, Leslie Ann Goldberg, Magnús M. Halldrsson, Anna Ingólfsdóttir, and Igor Walukiewicz, eds.), Lecture Notes in Computer Science, vol. 5125, Springer, 2008, pp. 370–384.

[7] James I. Lathrop, Jack H. Lutz, and Scott M. Summers, *Strict self-assembly of discrete Sierpinski triangles*, Theoretical Computer Science **410** (2009), 384–405.

[8] Ján Maňuch, Ladislav Stacho, and Christine Stoll, *Two lower bounds for self-assemblies at temperature 1*, ICBBE 2009: Bioinformatics and Biomedical Engineering , 2009. 3rd International Conference on, June 2009, pp. 1–4.

[9] ______, *Two lower bounds for self-assemblies at temperature 1*, SAC '09: Proceedings of the 2009 ACM Symposium on Applied Computing (Honolulu, Hawaii), ACM, 2009, pp. 808–809.

[10] Matthew J. Patitz and Scott M. Summers, *Self-assembly of decidable sets*, Natural Computing, to appear.

[11] Mojżesz Presburger, *Über die vollständigkeit eines gewissen systems der arithmetik ganzer zahlen*, welchem die Addition als einzige Operation hervortritt. Compte Rendus du I. Congrks des Mathematiciens des pays Slavs, Warsaw, 1930, pp. 92–101.

[12] Paul W. K. Rothemund, *Theory and experiments in algorithmic self-assembly*, Ph.D. thesis, University of Southern California, December 2001.

[13] Paul W. K. Rothemund and Erik Winfree, *The program-size complexity of self-assembled squares (extended abstract)*, STOC 2000: Proceedings of the Thirty-Second Annual ACM Symposium on Theory of Computing, 2000, pp. 459–468.

[14] Paul W.K. Rothemund, Nick Papadakis, and Erik Winfree, *Algorithmic self-assembly of DNA Sierpinski triangles*, PLoS Biology **2** (2004), no. 12, 2041–2053.

[15] Nadrian C. Seeman, *Nucleic-acid junctions and lattices*, Journal of Theoretical Biology **99** (1982), 237–247.

[16] Ryan Stansifer, *Presburger's article on integer arithmetic: Remarks and translation*, Tech. Report TR84-639, Cornell University, Computer Science Department, September 1984.

[17] Hao Wang, *Proving theorems by pattern recognition – II*, The Bell System Technical Journal **XL** (1961), no. 1, 1–41.

[18] ———, *Dominoes and the AEA case of the decision problem*, Proceedings of the Symposium on Mathematical Theory of Automata (New York, 1962), Polytechnic Press of Polytechnic Inst. of Brooklyn, Brooklyn, N.Y., 1963, pp. 23–55.

[19] Erik Winfree, *Algorithmic self-assembly of DNA*, Ph.D. thesis, California Institute of Technology, June 1998.