ECS 120: Theory of Computation
UC Davis — Vladimir Filkov

Section 1
November 3, 2005

Midterm

Instructions: Please answer the questions succinctly, thoughtfully and legibly. Good luck.

Name: SOLUTIONS

ID:

On section	you got	out of
1		10
2		15
3		30
4		45
5		45
6		30
Σ	CAP =	150

3 Justified True or False [30 points]

Put an **X** through the **correct** box. Provide a **brief** (but convincing) justification. No credit will be given to correct answers that lack a proper justification. Where appropriate, **make your justification a counter-example**. Each question in this section is worth 5 points.

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1. $(\emptyset \circ \emptyset)^* = \emptyset^0$

$\boxed{\cancel{\text{True}}}$ $\boxed{\text{False}}$

Explain:

$(\emptyset \circ \emptyset)^* = \emptyset^* = \{\varepsilon\}$

$\emptyset^0 = \{\varepsilon\}$

2. If L is not a regular language then L is context-free.

$\boxed{\text{True}}$ $\boxed{\cancel{\text{False}}}$

Explain:

$\{a^n b^n c^n \mid n \geq 0\}$ is neither context free nor regular

3. If CFG G_1 has fewer rules than CFG G_2 then $|L(G_1)| \leq |L(G_2)|$.

$\boxed{\text{True}}$ $\boxed{\cancel{\text{False}}}$

Explain:

G_1: $S_1 \to 0S_1$
$S_1 \to 1S_1$
$S_1 \to \varepsilon$

G_2: $S_2 \to T$
$T \to U$
$U \to V$
$V \to 0$

$L(G_1) = \Sigma^*$ $\Sigma = \{0,1\}$ $L(G_2) = \{0\}$

4. $L = \{a^i b^j | i \geq j \geq 65536\}$ is a regular language.

True **False**

Explain:

If L is regular than $a^k b^k$, where $k \geq \max(65536, \text{pump. length})$, must pump. But than $a^{k-1} b^k \in L$, which cannot be.

5. Any Context Free Grammar can be transformed into an equivalent grammar having one rule of the form $S \to V_1 V_2 \ldots V_n$ and n rules of the form $V_i \to a_i, i = 1, \ldots, n$, where V_i are variables and a_i are terminals.

True **False**

Explain:

A grammar of the form above can only derive strings no longer than n. Any grammar deriving strings of unbounded length cannot be represented like this

6. If $\overline{L}$ is finite then L is regular.

True **False**

Explain:

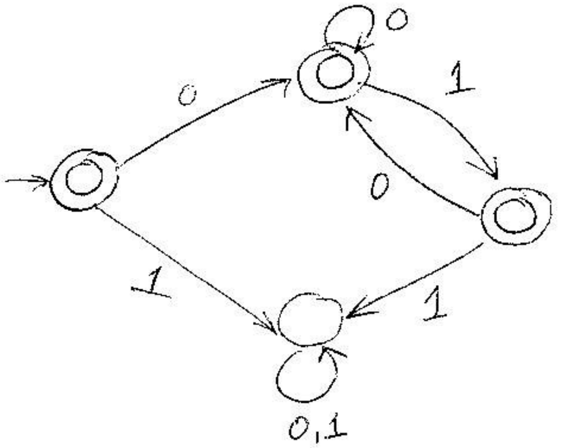
$\overline{L}$ is finite $\Rightarrow$

$\overline{L}$ is regular $\Rightarrow$

$\overline{\overline{L}} = L$ is regular

4 Finite Automata Problems [45 points]

1. (15 pts.) Give a DFA for the language of $(0 \cup 01)^*$.



$L((0 \cup 01)^*) = \{w \mid$ w starts on '0' and contains no '11' $\}$

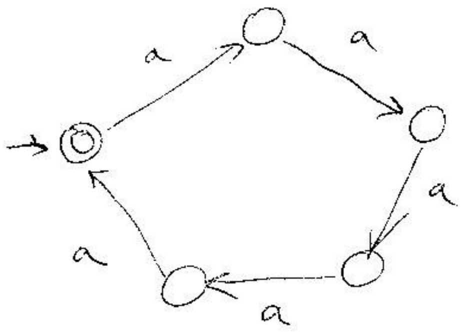
2. (30 pts.) Determine if the following languages are regular or not and prove your claim in each case.

a) $\{a^n b c^{3n} | n \geq 0\}$ Not Regular

Let $L = \{a^n b c^{3n} | n \geq 0\}$ be regular. Then if p is the pumping length, consider $w = a^p b c^{3p}$. Obviously $w \in L$ & $|w| > p$. Thus, $\exists x, y, z$ s.t. $w = xyz$ and $|y| > 0$, $|xy| \leq p$ & $xz \in L$. But then $y = a^i, p \geq i > 0$, so $xz = a^{p-i} b c^{3p}$, which cannot be in L.

Contradiction.

b) $\{a^{5n} | n \geq 0\}$ Regular



5 CFG Problems [45 pts.]

Write Context Free Grammars for the following languages:

a) (14 pts.) $A = \{ab^n c^m a^{n+m} c | n, m \geq 0\}$

$S_A \rightarrow aTc$
$T \rightarrow bTa \mid U \mid \varepsilon$
$U \rightarrow cUa \mid \varepsilon$

b) (12 pts.) $B = \{ab^n c^{n+m} a^m c | n, m \geq 0\}$

$S_B \rightarrow aVXc$
$V \rightarrow bVc \mid \varepsilon$
$X \rightarrow cXa \mid \varepsilon$

c) (7 pts.) $A \cup B$

$S_{A \cup B} \rightarrow S_A \mid S_B$ + all 6 rules from a) & b) above

d) (12 pts.) $A \cap B$

$A \cap B = \{a \quad c^m a^m c \mid m \geq 0\}$

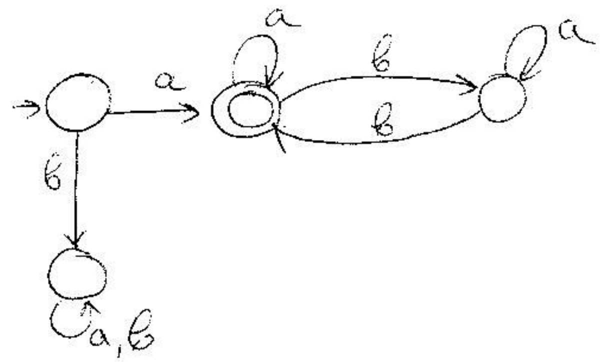
$S_{A \cap B} \longrightarrow a \quad Dc$

$D \longrightarrow cDa \mid \varepsilon$

(Hint: Consider the n's and the m's separately)

6 A Minimality Proof [30 points]

a) **(10 pts.)** Design a DFA for the language $\{w \in \{a,b\}^* | w$ starts with an a and contains an even number of b's$\}$.



b) **(20 pts.)** Find the smallest DFA for the language above and prove its minimality. (smaller=fewer states.)

Claim: The smallest DFA has 4 states.

Pf. Let $D = (Q, \Sigma, \delta, q_0, F)$ be a DFA for the language above s.t. $|Q| \leq 3$. Consider the strings $w_1 = \varepsilon$, $w_2 = b$, $w_3 = a$, $w_4 = ab$. Since D has fewer than 4 states, two of $\hat{\delta}(q_0, w_1), \hat{\delta}(q_0, w_2), \hat{\delta}(q_0, w_3), \hat{\delta}(q_0, w_4)$ must be the same state in D.

Case 1 If $\hat{\delta}(q_0, w_1) = \hat{\delta}(q_0, w_2)$
$\Rightarrow \hat{\delta}(q_0, a) = \hat{\delta}(q_0, ba)$, with $\hat{\delta}(q_0, a) \in F$ and $\hat{\delta}(q_0, ba) \notin F$. Contradict.

Case 2 If $\hat{\delta}(q_0, w_1) = \hat{\delta}(q_0, w_3)$, with $\hat{\delta}(q_0, w_1) \notin F$ and $\hat{\delta}(q_0, w_3) \in F$. Contradiction

Case 3 If $\hat{\delta}(q_0, w_1) = \hat{\delta}(q_0, w_4)$
$\Rightarrow \hat{\delta}(q_0, ab) = \hat{\delta}(q_0, abab)$, with $\hat{\delta}(q_0, ab) \notin F$ and $\hat{\delta}(q_0, abab) \in F$. Contr.

Case 4 If $\hat{\delta}(q_0, w_2) = \hat{\delta}(q_0, w_3)$, with $\hat{\delta}(q_0, w_2) \notin F$ and $\hat{\delta}(q_0, w_3) \in F$. Contradict.

Case 5 If $\hat{\delta}(q_0, w_2) = \hat{\delta}(q_0, w_4)$
$\Rightarrow \hat{\delta}(q_0, bb) = \hat{\delta}(q_0, abb)$, with $\hat{\delta}(q_0, bb) \notin F$ and $\hat{\delta}(q_0, abb) \in F$. Contrad.

Case 6 If $\hat{\delta}(q_0, w_3) = \hat{\delta}(q_0, w_4)$, with $\hat{\delta}(q_0, w_3) \in F$ and $\hat{\delta}(q_0, w_4) \notin F$. Contr.

$\Rightarrow \boxed{|Q| \geq 4}$

1 Definitions [10 points]

1. (4 pts.) Define formally the star (*) operation on languages, i.e. $A^* = \ldots$

$A, B \subseteq \Sigma^*$; $x, y \in \Sigma^*$

$A \circ B = \{xy \mid x \in A, y \in B\}$, $A^i = \begin{cases} \{\varepsilon\}, & i = 0 \\ A \circ A^{i-1}, & i > 0 \end{cases}$, $A^* = \bigcup_{i=0}^{\infty} A^i$

2. (6 pts.) State the Pumping Lemma for regular languages (formally and completely).

If L is a regular language then $\exists p > 0$, such that $\forall w \in L$ for which $|w| \geq p$, w can be represented as $w = xyz$ where

(1) $|y| > 0$

(2) $|xy| \leq p$

(3) $xy^iz \in L$, $i \geq 0$

2 A Decision Procedure [15 pts.]

You are given a DFA D and want to determine if it accepts any words of which your name is a substring. Describe a decision procedure to do that.

DP1 $w = w_1 w_2 \ldots w_n$, is your name

1. $L = \Sigma^* \circ \{w\} \circ \Sigma^*$
2. Design an NFA N, s.t. $L = L(N)$ (as concatenation of three languages)
3. Convert N to a DFA
4. Construct a DFA for $L(DFA(N)) \cap L(D)$
5. Check if the DFA from step 4. accepts no strings. (emptiness problem).

DP2 $w = w_1 w_2 \cdots w_n$, is your name

1. For each state q_i of D, $q \in \{q_1, q_2, \cdots q_n\}$
2. Let $q' = \hat{\delta}(q_i, w)$
3. If an accept state is reachable from q', your name is a substring