ECS 20 **Chapter 3, Functions and Algorithms**

1. Introduction
    1.1. Functions "map" one object to another object. The objects can be anything, e.g. numbers, sets, or cities. We will concentrate on integers.
    1.2. Algorithms are finite, step-by-step, lists of well defined steps to solve a problem. The problem can be from any subject, but again we will concentrate on integers.
2. Functions
    2.1. Definition: A function f from a set X to a set Y, denoted $f : X \to Y$, is a relation from X, the "domain", to Y, the "range," that has every element in X related to exactly one element in Y. Such relations may be represented as a series of ordered pairs, (a, b), where $a \in X$, and $b \in Y$.
    2.2. If the sets of the function are small, then arrow diagrams can be quite useful.
    2.3. Examples. Given the domain $D = \{3, 4, 5\}$, and the range $R = \{1, 3, 5, 7\}$, which relations are functions?
        2.3.1. $g = \{(3, 3), (5, 1), (4, 3)\}$, is a function.
        2.3.2. $h = \{(3, 1), (5, 7)\}$, is not a function because 4 of the domain is not mapped.
        2.3.3. $j = \{(3, 1), (4, 3), (5, 5), (3, 7)\}$, is not a function because 3 of the domain is mapped more than once.
        2.3.4. $f(x) = \sqrt{x}$ is a function for the domain positive real numbers $\mathbf{R}^+$, and the range $\mathbf{R}^+$. It passes the "vertical line" test, since no vertical line intersects the graph of the function at more than one point.
        2.3.5. $f(x) = \sqrt{x}$ is not a function for the domain positive real numbers $\mathbf{R}^+$, and the range of all real numbers $\mathbf{R}$, because each a in $\mathbf{R}^+$ maps to two numbers in $\mathbf{R}$, $|\sqrt{x}|$ and $-|\sqrt{x}|$. For example, the vertical line for $x = 4$, would pass through (4, 2) and (4, -2).
3. One-to-One, Onto, and Invertible Functions
    3.1. One-to-One
        3.1.1. Definition: A function f is "one-to-one" when no two elements in the domain map to the same element in the range. Symbolically, $f: X \to Y$ is one-to-one $\leftrightarrow \forall x_1, x_2 \in X$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$. Note that the function need not map to every element in the range.
        3.1.2. Such functions pass the "horizontal line" test, since no horizontal line, $f(x)$ value, intersects the graph of the function at more than one point.
        3.1.3. Examples:
            3.1.3.1. Define $f: \mathbf{R} \to \mathbf{R}$ and $f(x) = 3x - 4$. Prove f is one-to-one.

We must show $\forall x_1, x_2 \in \mathbf{R}$, if $3x_1 - 4 = 3x_2 - 4$, then $x_1 = x_2$. This is a simple direct proof. The following proof is a bit pedantic because it shows each step of the algebraic manipulation, but it is the correct form.

Assertion	Reason
1. $3x_1 - 4 = 3x_2 - 4$	premise
2. $3x_1 = 3x_2$	(1) subtract 4 from both sides.
3. $x_1 = x_2$	(2) divide by 3 on both sides. QED

            3.1.3.2. Define $g: \mathbf{Z} \to \mathbf{Z}$ and $g(x) = x^2$. Prove g is not one-to-one.

We must show $\neg\forall x_1, x_2 \in X$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$.
By De Morgans, this is equivalent to $\exists x_1, x_2 \in X$, if $f(x_1) = f(x_2)$, then $x_1 \neq x_2$.
Existential statements can best be proved by simply providing an example.
For this problem we must show $\exists x_1, x_2 \in \mathbf{Z}$, if $g(x_1) = g(x_2)$, then $x_1 \neq x_2$.
Let $x_1 = 3$, and $x_2 = -3$, then $g(x_1) = g(3) = 3^2 = 9$, and $g(x_2) = g(-3) = (-3)^2 = 9$.
Hence, $(x_1) = g(x_2)$, but $x_1 \neq x_2$, so g is not one-to-one.

        3.1.4. Computer Application: Hash functions

Hash tables are a data structure that to store data into arrays that are much smaller than the possible range of values of the data. Computer programs use hash tables because they are the fastest data structures for finding data. A hash function converts from the values of the data to a range of values equal to the size of the array. For example, a hash table could be used to store the information about UCD students based on their SIDs. Since there are nine digits in a SID, there are 10^9 possible SIDs, but there are only 30,000 students. If we had an array of size

100,000, we could use the hash function SID mod 100000 to convert any SID to an index within the array. (Mod is short for modulo, which is the remainder of an integer division.) For example, 999428182 mod 100000 would be placed in array position 28182. By their nature, hash functions are not one-to-one, e.g. 555528182 mod 100000 would also be placed in array position 28182. Such "collisions" may be handled by moving from the calculated position in the array until an empty position is found. These collision resolution methods combine with the hash function to create an efficient one-to-one system that guarantees that if an item is not found using the system, then the item is not in the array.

3.2. Onto Definition: A function f is "onto" if there is at least one element in the domain that maps to every element in the range. Symbolically, f: $X \to Y$ is onto $\leftrightarrow \forall y \in Y, \exists x \in X$ such that $f(x) = y$. Note that more than one element in the domain may map to the same element in the range.

3.2.1. Examples:

3.2.1.1. if f: $\mathbf{R} \to \mathbf{R}$ and $f(x) = 3x - 4$. Prove f is onto.

We must show that if f: $\mathbf{R} \to \mathbf{R}$ and $y \in \mathbf{R}$ and $f(x) = 3x - 4$, then $\exists x \in \mathbf{R}$ such that $f(x) = y$.

This means we must show both that there exists $a \in \mathbf{R}$ and that $f(a) = y$ for any given y.

Assertion	Reason
1. $y \in \mathbf{R}$	premise
2. r is an instance of y	(6) by universal instantiation
3. Let $a = \frac{r+4}{3}$	(2) by variable instantiation
4. $a \in \mathbf{R}$	(2) and (3) and that the sum and quotients (other than dividing by 0) of real numbers are real numbers.
5. $f(x) = 3x - 4$	premise
6. $f(a) = 3a - 4$	(5) by definition of function
7. $f(a) = 3\frac{r+4}{3} - 4$	(3) and (6) by substitution
8. $f(a) = r$	(7) and algebra
9. $\exists x \in \mathbf{R}$ such that $f(x) = r$	(4) and (8) by existential generalization
10. $\forall y \in \mathbf{R}, \exists x \in \mathbf{R}$ such that $f(x) = y$	(2) and (9) by universal generalization. QED

3.2.1.2. if f: $\mathbf{Z} \to \mathbf{Z}$ and $f(x) = 3x - 4$. Prove f is not onto.

We want to prove $\neg\forall y \in \mathbf{Z}, \exists x \in \mathbf{Z}$ such that $3x - 4 = y \equiv \exists y \in \mathbf{Z}, \exists x \in \mathbf{Z}$ such that $3x - 4 \neq y$.

As before, proving the negation of a universal, requires only one counterexample. Since the function only produces every third integer, there an infinite number from which to choose, i.e. $3x$ -3 and $3x - 2$ are not mapped. For example, let $x = 5$, then $3x - 2 = 15 - 2 = 13$. Then if $f(x) = 3x - 4 = 13$, we have $x = \frac{13+4}{3} = \frac{17}{3}$, which is not an integer. QED

3.3. Invertible = a function is "invertible" if its inverse relation is also a function, i.e. $f(x)$: $X \to Y$, then $f^{-1}(y)$: $Y \to X$.

3.3.1. A function is invertible if and only if it is both one-to-one and onto. That is, every element in the domain is mapped to exactly one element in the range, and all of the elements in the range are paired. Compare the arrow diagrams.

3.3.2. Example. We've shown that $f(x) = 3x - 4$ is both one-to-one and onto. What is its inverse function, $f^{-1}(y)$? Since $f(x) = 3x - 4 = y$, then $x = f^{-1}(y) = \frac{y+4}{3}$

4. Mathematical Functions, Exponential and Logarithmic Functions

4.1. Floor, $\lfloor x \rfloor$ denotes the greatest integer that does not exceed x, e.g. $\lfloor 7.5 \rfloor = 7$. With $\mathbf{R} \to \mathbf{Z}$, is it one-one, onto, invertible?

4.2. Ceiling, $\lceil x \rceil$ denotes the least integer that is not less than x, e.g. $\lceil -7.3 \rceil = -7$, $\mathbf{R} \to \mathbf{Z}$. Is it one-one, onto, invertible?

4.3. INT(x) converts x by deleting the fractional part of the number, e.g. INT(5.3) = 5, INT(-4.7) = -4. With $\mathbf{R} \to \mathbf{Z}$, is it one-one, onto, invertible?

4.4. Absolute value, $|x|$ is the greater of $-x$, and x, $\mathbf{R} \to \mathbf{R}$. If $x < 0$, then $|x| = -x$, and if $x \geq 0$, then $|x| = x$. For $\mathbf{Z} \to \mathbf{Z}$, is it one-one, onto, invertible?

4.5. Remainder, $x \bmod M$ denotes the integer remainder from dividing x by M, $\mathbf{Z} \to \mathbf{N}$, $M \neq 0$. If $x \geq 0$, then x mod M = r, where $x = Mq + r$ and $q \in \mathbf{N}$ and $0 \leq r < M$. If $x < 0$, then x mod $M = r$, where $-x = Mq + r'$, $q \in \mathbf{N}$, $0 \leq r' < M$, if $r' \neq 0$ then $r = M - r'$ else $r = 0$. For a given $M \neq 0$, what is the range of values for x mod M, is it one-one, onto, invertible?

4.5.1. Examples: 17 mod 5 = 2, -15 mod 9 = 9 – 6 = 3, -24 mod 6 = 0.

4.6. Exponential functions for all real numbers $a^x = \lim_{r \to x} a^r$, $\mathbf{R} \to \mathbf{R}$. If $x = \frac{m}{n}$, where $m,n \in \mathbf{Z}$, then $a^x = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$, e.g. $36^{3/2} = \sqrt[2]{36^3} = \sqrt[2]{2*2*2*2*2*2*3*3*3*3*3*3} = 2*2*2*3*3*3 = 216 = \left(\sqrt[2]{36}\right)^3$ Is a exponential function of the form $f(x) = a^x$ one-one, onto, invertible?

4.7. Logarithmic functions represent the exponent to which a base b, must be raised to obtain x, where x, $b \neq 0$. $y = \log_b x$, then $x = b^y$, e.g. if $3 = \log_5 125$, then $125 = 5^3$.

4.7.1. Logarithmic functions are the inverse of exponential functions. So if $f(x) = \log_b x$, then $f^{-1}(y) = b^x$. Are they invertible for $\mathbf{R} \to \mathbf{R}$, $\mathbf{Z} \to \mathbf{Z}$, $\mathbf{N} \to \mathbf{N}$ for any x, $b \neq 0$?

4.7.2. Example, if $f(x) = \log_7 x$, then $f^{-1}(y) = 7^y$, so $f(49) = \log_7(49) = 2$, and $f^{-1}(2) = 7^2 = 49$.

5. Sequences, Indexed Classes of Sets

5.1. A "sequence" is a function whose domain is either all the integers between two give integers, or all the integers greater than or equal to a given integer. Typically represented as a set of elements written in a row, e.g. $a_m, a_{m+1}, a_{m+2}, \ldots, a_{m+n-1}, a_{m+n}$.

5.1.1. In computer programming, for-loops and arrays are implementations closely linked to sequences.

5.2. Summation $\sum_{i=1}^{n} i = 1 + 2 + 3 + \ldots + n - 1 + n = \frac{n(n+1)}{2}$

5.3. Use induction to prove $\sum_{i=1}^{n} i = 1 + 2 + 3 + \ldots + n - 1 + n = \frac{n(n+1)}{2}$

Let $S_n = \sum_{i=1}^{n} i$

Basis: $P(1) = S_1 = \sum_{i=1}^{1} i = 1$, $\frac{n(n+1)}{2} = \frac{1(1+1)}{2} = \frac{2}{2} = 1$, QED

Inductive Hypothesis: $P(k) = S_k = \sum_{i=1}^{k} i = \frac{k(k+1)}{2}$

Inductive Conclusion: $P(k+1) = S_{k+1} = \sum_{i=1}^{k+1} i = \frac{(k+1)(k+1+1)}{2}$

Assertion	Reason
$S_{k+1} = 1 + 2 + 3 + \ldots + k + k + 1$	Inductive Conclusion and definition of summation
$= S_k + k + 1$	Definition of S
$= \frac{k(k+1)}{2} + k + 1$	By inductive hypothesis
$= \frac{k(k+1) + 2(k+1)}{2}$	
$= \frac{(k+1)(k+2)}{2}$	
$= \frac{(k+1)(k+1+1)}{2}$	This is what to be shown.

With the basis and the hypothetical conclusion both being shown true, the statement is proved by induction.

6. Recursively Defined Functions are functions that refer to, i.e. call, themselves.

6.1. In order to work properly a recursive function must have two properties:

6.1.1. There must be certain arguments for which the function does not refer to itself. Think of this as $P(0)$ in induction.

6.1.2. Each time the function does refer to itself, the argument of the function must be closer to a base value. Think of this as the inductive steps of induction.

6.2. Example: n factorial $= n(n-1)(n-2)\ldots 3 * 2 * 1 = n!$

A function to compute the *nth* factorial could be:

if $n = 0$, then $n! = 1$

if $n > 0$, then $n! = n * (n-1)!$

7. Cardinality = size of a set, versus an ordinal number that refers to the order of an element in a sequence, e.g. eighth.

7.1. The set $\mathbf{Z}^+$ of counting numbers {1, 2, 3, …} is the most basic infinite set. A set is called "countably infinite", if and only if it has the same cardinality as the set of positive integers $\mathbf{Z}^+$. A set is called "countable" if, and only if, it is finite or countably infinite.

7.2. Show that set **Z** is countable.

The set **Z** of all integers is infinite, so to countable it must be countably infinite. To show that **Z** is countably infinite we must find a function from **Z+** to **Z** that is one-to-one and onto.

f(x) = if *x* is an even then $\frac{x}{2}$, if *x* is an odd then $-\frac{x}{2}$

8. Algorithms

8.1. Algorithm = a finite step-by-step list of well-defined instructions for solving a particular problem. The choice of algorithm often depends on their efficiency.

8.2. Example. Find an efficient method to evaluate a polynomial of degree *d*, e.g. a *d* = 4 polynomial could be $f(x) = 3x^4 + 7x^3 + 4x^2 + 12x + 6$.

8.2.1. Direct algorithm treats each term separately, so *f(x)* = 3 * *x* * *x* * *x* * *x* + 7 * *x* * *x* * *x* + 4 * *x* * *x* + 12 * *x* + 6. This in involves 1 + 2 + 3 … + *d* multiplications, and *d* additions, so $\left(\frac{d(d+1)}{2} + d\right)$ operations.

8.2.2. Horner's Method factors out the *x* to minimize the computation of the powers of *x*, so *f(x)* = (((((((3 * *x*) + 7) * *x*) + 4) * *x*) + 12) * *x*) + 6, which takes *d* multiplications and *d* additions, so 2*d* operations.

9. Complexity of Algorithms = measures of the <u>relative</u> efficiency of algorithms with regard to time, and sometimes space.

9.1. Definitions of functions to establish the relative rates of growth of functions.

9.1.1. T(N) = O(f(N)) if there are c > 0 and n > 0 such that T(N) ≤ cf(N) when N ≥ n.

9.1.1.1. $T(N) = N^2 + 25 = O(N^2)$

9.1.2. T(N) = Ω (f(N)) if there are c > 0 and n > 0 such that T(N) ≥ cf(N) when N ≥ n.

9.1.3. T(N) = Θ(f(N)) if and only if T(N) = O(f(N)) and T(N) = Ω (f(N)).

9.1.4. What order is an algorithm that has a growth-rate function

9.1.4.1. Linear search = N =

9.1.4.2. $8 * N^3 + 9 * N <= c * N^3$ where c >= 9 and N >= 1

9.1.4.3. $7 * \log_2 N + 20 <= c * \log_2 N$ where c >= 8 and $N >= 2^{20}$;

9.1.4.4. $7 * \log_2 N + N <= c * N$ where c = 8 and N >= 0

9.2. Rules

9.2.1. If T(N) = O(f(N)) and V(N) = O(g(N)) then

9.2.1.1. T(N) + V(N) = max(O(f(N)), O(g(N)))

9.2.1.2. T(N) * V(N) = O(f(N) * g(N))

9.2.2. If T(N) is a polynomial of degree k, then $T(N) = \Theta(N^k)$. This means you ignore lower-order terms and multiplicative constants.

9.2.2.1. Show that any polynomial $f(x) = c_k x^k + c_{k-1} x^{k-1} + \ldots + c_1 x + c_0$ is $O(x^k)$

$$
\begin{aligned}
f(x) &\le \sum_{i=0}^{k} |c_i| x^i \\
&\le x^k \sum_{i=0}^{k} |c_i| x^{i-k} \\
&\le x^k \sum_{i=0}^{k} |c_i|,\ for\ x \ge 1 \\
&\le c_T x^k,\ for\ x \ge 1\ where\ c_T = \sum_{i=0}^{k} |c_i|
\end{aligned}
$$

9.2.3. $\log^k N = O(N)$ for any constant k. This tells us that logarithms grow slower than linear time.

9.3. Style

9.3.1. Don't include constants or low order terms inside a Big-Oh.

9.3.2. Don't say T(N) ≤ O(f(N)). The inequality is implied by the definition.