ECS289F — Homework 1 Solutions

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Problem 1. *We say that σ-structures $\mathfrak{A} = \langle A, \{P_i^{\mathfrak{A}}\}, \{f_i^{\mathfrak{A}}\}, \{c_i^{\mathfrak{A}}\}\rangle$ and $\mathfrak{B} = \langle B, \{P_i^{\mathfrak{B}}\}, \{f_i^{\mathfrak{B}}\}, \{c_i^{\mathfrak{B}}\}\rangle$ are* isomorphic *if there exists a bijective mapping $h : A \to B$ such that*

- *for each k-ary predicate symbol P_i in σ and $\bar{a} \in A^k$, we have $\bar{a} \in P_i^{\mathfrak{A}}$ iff $h(\bar{a}) \in P_i^{\mathfrak{B}}$;*
- *for each k-ary function symbol f_i in σ and $\bar{a} \in A^k$, we have $h(f_i^{\mathfrak{A}}(\bar{a})) = f_i^{\mathfrak{B}}(h(\bar{a}))$; and*
- *for each constant symbol c_i in σ, we have $h(c_i^{\mathfrak{A}}) = c_i^{\mathfrak{B}}$.*

Show that if $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic, then for any FO formula $\varphi(x_1, \ldots, x_n)$ and $\bar{a} \in A^k$, we have $\mathfrak{A} \models \varphi(\bar{a})$ iff $\mathfrak{B} \models \varphi(h(\bar{a}))$. (In other words, FO is generic*.)*

Solution. By induction on terms and formulae, we show that for any term t, $h(t^{\mathfrak{A}}(\bar{a})) = t^{\mathfrak{B}}(h(\bar{a}))$, and that for any formula φ, we have $\mathfrak{A} \models \varphi(\bar{a})$ iff $\mathfrak{B} \models \varphi(h(\bar{a}))$. In the base case:

- If t is a constant symbol c, then $h(t^{\mathfrak{A}}(\bar{a})) = h(c^{\mathfrak{A}}) = c^{\mathfrak{B}} = t^{\mathfrak{B}}(h(\bar{a}))$ as required.
- If t is a variable x_i, then $h(t^{\mathfrak{A}}(\bar{a})) = h(a_i) = t^{\mathfrak{B}}(h(\bar{a}))$.

In the inductive case, we assume that the claim holds for immediate sub-terms and sub-formulae.

- If $t = f(t_1, \ldots, t_k)$, we have

$$\begin{array}{rcll} h(t^{\mathfrak{A}}(\bar{a})) & = & h(f^{\mathfrak{A}}(t_1^{\mathfrak{A}}(\bar{a}), \ldots, t_k^{\mathfrak{A}}(\bar{a}))) & \\ & = & f^{\mathfrak{B}}(h(t_1^{\mathfrak{A}}(\bar{a})), \ldots, h(t_k^{\mathfrak{A}}(\bar{a}))) & \text{By assumption} \\ & = & f^{\mathfrak{B}}(t_1^{\mathfrak{B}}(h(\bar{a})), \ldots, t_k^{\mathfrak{B}}(h(\bar{a}))) & \text{By inductive hypothesis} \\ & = & t^{\mathfrak{B}}(h(\bar{a})) & \end{array}$$

  as required.

- If φ is $t_1 = t_2$, then we have

$$\begin{array}{rcll} \mathfrak{A} \models \varphi(\bar{a}) & \text{iff} & t_1^{\mathfrak{A}}(\bar{a}) = t_2^{\mathfrak{A}}(\bar{a}) & \\ & \text{iff} & h(t_1^{\mathfrak{A}}(\bar{a})) = h(t_2^{\mathfrak{A}}(\bar{a})) & \text{Since } h \text{ is bijective} \\ & \text{iff} & t_1^{\mathfrak{B}}(h(\bar{a})) = t_2^{\mathfrak{B}}(h(\bar{a})) & \text{By inductive hypothesis} \\ & \text{iff} & \mathfrak{B} \models \varphi(h(\bar{a})) & \end{array}$$

  as required.

- If φ is $P(t_1, \ldots, t_n)$, then we have

$$\begin{array}{rcll} \mathfrak{A} \models \varphi(\bar{a}) & \text{iff} & (t_1^{\mathfrak{A}}(\bar{a}), \ldots, t_n^{\mathfrak{A}}(\bar{a})) \in P^{\mathfrak{A}} & \\ & \text{iff} & (h(t_1^{\mathfrak{A}}(\bar{a})), \ldots, h(t_n^{\mathfrak{A}}(\bar{a}))) \in P^{\mathfrak{B}} & \text{By assumption} \\ & \text{iff} & (t_1^{\mathfrak{B}}(h(\bar{a})), \ldots, t_n^{\mathfrak{B}}(h(\bar{a}))) \in P^{\mathfrak{B}} & \text{By inductive hypothesis} \\ & \text{iff} & \mathfrak{B} \models \varphi(\bar{a}) & \end{array}$$

  as required.

- If φ is $\varphi_1 \wedge \varphi_2$, then we have

$$\begin{array}{lll} \mathfrak{A} \models \varphi(\bar{a}) & \text{iff} & \mathfrak{A} \models \varphi_1(\bar{a}) \text{ and } \mathfrak{A} \models \varphi_2(\bar{a}) \\ & \text{iff} & \mathfrak{B} \models \varphi_1(h(\bar{a})) \text{ and } \mathfrak{B} \models \varphi_2(h(\bar{a})) \quad \text{By inductive hypothesis} \\ & \text{iff} & \mathfrak{B} \models \varphi(h(\bar{a})) \end{array}$$

as required. (The cases for $\vee$ and $\neg$ are similar.)

- If φ is $\exists x\ \varphi_1(x)$, then we have

$$\begin{array}{lll} \mathfrak{A} \models \varphi(\bar{a}) & \text{iff} & \mathfrak{A} \models \varphi_1(a', \bar{a}) \text{ for some } a' \in A \\ & \text{iff} & \mathfrak{B} \models \varphi_1(h(a'), h(\bar{a})) \qquad \text{By inductive hypothesis} \\ & \text{iff} & \mathfrak{B} \models \varphi(\bar{a}) \end{array}$$

as required. (The case for $\forall$ is similar.) □

Problem 2. *Give an example of an FO sentence that is finitely valid but not valid. (You may assume any vocabulary σ that you wish.)*

Solution. One such example is the FO sentence φ that says, "if $\leq$ is a linear order over a non-empty domain, then there is a smallest element" (where the vocabulary consists of a single binary predicate $\leq$):

$$\begin{array}{rlll} \varphi & \stackrel{\text{def}}{=} & ((\forall x\ \forall y\ (x \leq y \wedge y \leq x) \rightarrow x = y) & \text{(antisymmetry)} \\ & & \quad \wedge\ (\forall x\ \forall y\ \forall z\ (x \leq y \wedge y \leq z) \rightarrow x \leq z) & \text{(transitivity)} \\ & & \quad \wedge\ (\forall x\ \forall y\ x \leq y \vee y \leq x) & \text{(totality)} \\ & & \quad \wedge\ (\exists x\ x = x)) & \text{(non-emptiness)} \\ & & \rightarrow (\exists x\ \forall y\ x \leq y) & \end{array}$$

(Thanks to Daniel for pointing out the non-emptiness requirement.) □

Problem 3. *We say that relational calculus queries Q, Q' are* equivalent *if for every database instance I, we have $[\![Q]\!]^I = [\![Q']\!]^I$. Show that this problem is undecidable.*

Solution. By reduction from checking finite validity of FO sentences, which is undecidable by Trakhtenbrot's Theorem. Given FO sentence φ, we construct Boolean (0-ary) relational calculus queries Q, Q' as follows:

$$\begin{array}{rcl} Q & \stackrel{\text{def}}{=} & \{() \mid \varphi\} \\ Q' & \stackrel{\text{def}}{=} & \{() \mid \mathsf{true}\} \end{array}$$

where true is shorthand for your favorite tautology, e.g., $c = c$ where c is a constant symbol. Note that

$$\begin{array}{rcl} [\![Q]\!]^I & = & \begin{cases} \{()\} & \text{if } I \models \varphi \\ \emptyset & \text{otherwise} \end{cases} \\ [\![Q']\!]^I & = & \{()\} \quad \text{for any } I \end{array}$$

and hence it is clear that Q and Q' are equivalent iff $\varphi \in$ FIN-VALID. □