1 Elimination Thresholds in general games

In the general setting there is, as before, a history of games already played, and a schedule of remaining games. But now, each game results in one of a finite set of payoffs for the two teams. A finite set of payoffs is called a "scoring system". For example, in baseball, the scoring system is that the winner gets one point and the loser gets zero. In contrast to baseball, for games where ties are allowed, one scoring system is that one point goes to each team when a tie occurs, and two points goes to the winning team, and none to the losing team, when a win occurs (this is the scoring system for in the National Hockey League. In another scoring system a winner might get three points and the loser gets zero, while a tie gives one point to each team. This is the scoring system used for European Football. It was recently shown that the elimination question in this scoring system is NP-Complete. We represent a scoring system by the finite set of possible payoffs. Thus, the baseball scoring system is represented as (1,0); the hockey scoring system is represented as (1,1), (2,0); and the European football (soccer) scoring system is represented as (1,1), (3,0).

Abstractly, a scoring system is a finite list of payoff pairs, and the outcome of any game is associated with exactly one payoff pair in that list. If the outcome of a game is associated with the specific payoff pair (x, y), then one team gets x points and the other gets y points. The league leader at the end of the season, is the team with the most total points. The problem is again to determine which teams have been eliminated, i.e., cannot strictly win or tie for the most points, under *any* scenario for the remaining games. In this setting, let $w(i)$ be the points team i has won so far, and let $g(i)$ be the maximum possible points that team i could get from it's remaining games, and define $m(i) = w(i) + g(i)$, i.e., the maximum possible points that team i could end up with.

For the following result, we use a *monotonicity* restriction on the scoring system.

Let p_{max} be the *best* possible payoff in any payoff to a team in the scoring system. For example, in the case of soccor, $p_max = 3$. The monotonicity restriction on the scoring system is:

If (x, y) is any of the possible payoffs in the scoreing system, where $x < p_{max}$, then there is also payoff in the scoring system (x_{max}, y') where $y' \leq y$.

Clearly, the three scoring systems mentioned above, (1,0) for baseball; (2,0), (1,1) for hockey; and (3,0),(1,1) for soccer, obey this restriction.

THEOREM: For any scoring system where the monotonicity restriction holds, there is a number W^* such that any team i is eliminated if and only if $W(i) < W^*$.

PROOF: Let S denote a selection of outcomes for the remaining games (a scenario for the remaining games), and let $W(S)$ denote the maximum number of points (from games already played and from the remaining games) won by any team under scenario S. Over all possible scenarios, define S^* as a scenario S where $W(S)$ is minimum. Define W^* to be $W(S^*)$.

Consider any team i. If $W(i) < W(S^*)$, then by the definition of $W(S^*)$, team i is eliminated in all scenarios, proving the if direction of the theorem.

Convesely, assume $W(i) \geq W(S^*)$. If i is not the leader (either strict or tied) in scenario S^*, then modify scenario S^* so that i receives a total of $g(i)$ points from its remaining games (so i gets the best possible outcome $p_m ax$ from each remaining game. For each remaining game (i, j) that i plays, which had a payoff of (x, y) in S^* (x to i, and y to j), change the scenario by making the new payoff $(p_{max}, y'\ )$, where $y' \leq y$. By the monotonicity restriction, this is always possble. Thus team i now has $W(i)$ points in the new scenario, no other team receives more points than it received under S^*, so under the new scenario, team i is either the undisputed leader, or ties for the lead. This proves the only if direction of the Theorem.

Note that the above proof has no connection to network flow, cuts or linear programming, showing that the general threshold phenomenon is not inherently related to these structures, or to any other structures used to compute the thresholds. Note also that Theorem **??** has easy extensions to games played between more than two teams at a time, provided the natural generalization of monotonicity holds.