Homework 3, Problem 3: The Node Cover Problem

Let $G = (V, E)$ be an undirected bipartite graph with node weights given by the function $w \colon V \to]0, \infty[$. Then there is a partition of V into two disjoint nonempty subsets V_1, V_2. We form a new undirected graph G' by adding two nodes s, t to G and connecting every node in V_1 with s and every node in V_2 with t. We define edge weights as follows:

$$c(x, y) = \begin{cases} w(y) & \text{if } x = s \\ w(x) & \text{if } y = t \\ \infty & \text{otherwise} \end{cases}$$

(If the weight ∞ looks fishy to you, just take a number larger than the sum of all node weights instead.) If we cut *all* edges of the form (s, v_1) with $v_1 \in V_1$ and (v_2, t) with $v_2 \in V_2$, then this is a valid (s,t)-cut with capacity equal to the sum over all $w(i)$. Therefore a *minimal* cut of G' will necessarily cut only edges of the form (s, v_1) with $v_1 \in V_1$ and (v_2, t) with $v_2 \in V_2$, because including any other edge would result in a larger cut.

Given a minimum cut $S, T \subseteq V$ of G' such that $s \in S$, $t \in T$, let N be the set of all nodes v in V such that v is incident to an edge that is cut. We claim that N is a node cover with minimal weight. If (v_1, v_2) is an edge in G, then without loss of generality $v_1 \in V_1$ and $v_2 \in V_2$, so that there is a path (s, v_1, v_2, t) in G'. Since $c(v_1, v_2) = \infty$ it is not cut, so one of the two edges (s, v_1), (v_2, t) must be cut. Therefore at least one of v_1, v_2 is in N. By definition of the weight function c, the weight of the node cover N is the capacity of the cut (S, T). Conversely, if N is a node cover of G, then let we can form a – not necessarily minimal – (s,t)-cut of G' by cutting all edges (s, x) resp., (x, t) such that $x \in N$. This is a cut because for any path (s, x, y, t), the edge (x, y) must be incident to some node in N, hence (s, x) or (y, t) is cut. Similarly, the capacity of the constructed cut equals the weight of the node cover N by definition of the weight function c.

We conclude that a minimal (s,t)-cut in G' corresponds to minimal node cover: If it would not, there was a smaller node cover, which would correspond to a cut with a value smaller than that of the minimal cut.