Bernd Hamann 3/19/2018

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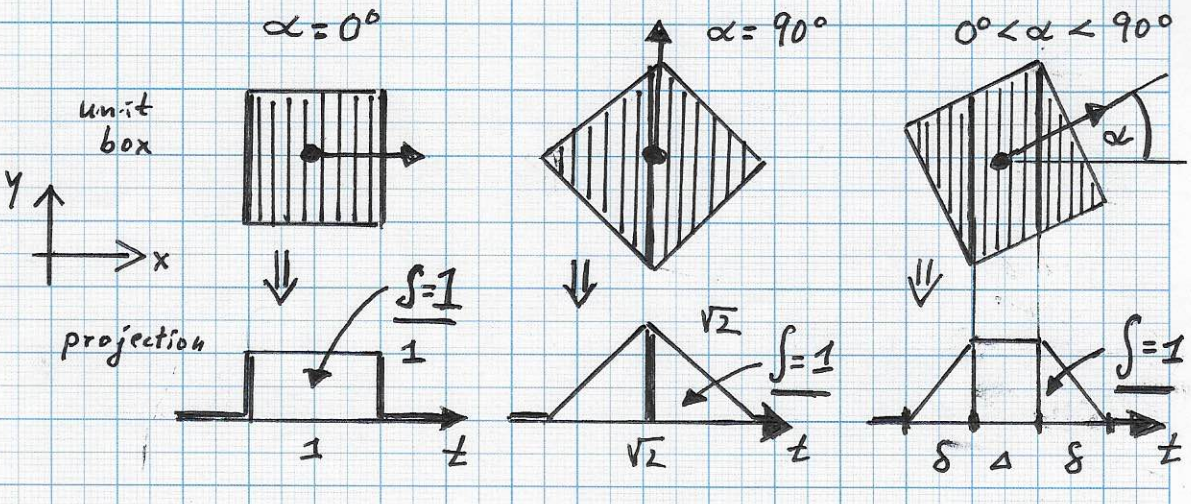
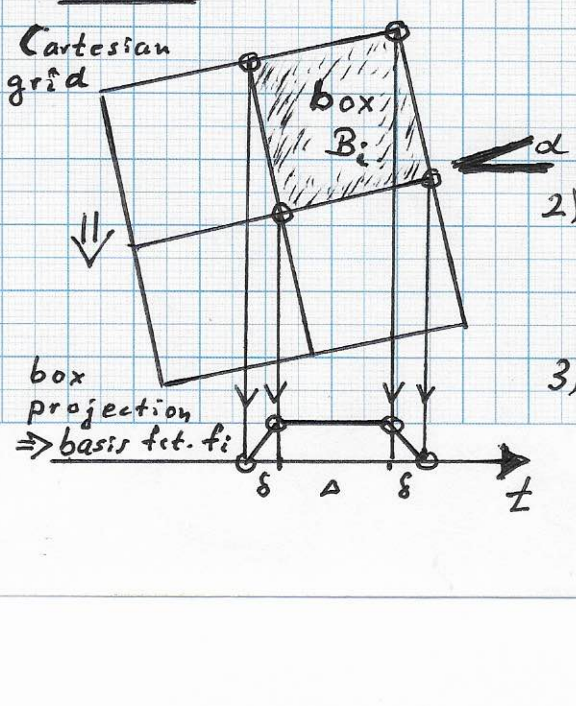
■ Towards OPTIMAL Reconstruction Using a SIMPLE Algorithm with MINIMAL Time Complexity

→ **Background:** "Projections" of a BOX (= square pixel) onto a line; orientation of box: α



type • CONSTANT ($\delta = 0$, $\Delta = 1$) • LINEAR ($\delta = \sqrt{2}/2$, $\Delta = 0$) • CONSTANT+LINEAR (general case)

→ **IDEA:**



1) The function to be reconstructed in 2D space is defined as a piecewise constant function over a Cartesian grid.

2) The (unknown) bivariate function on the Cartesian grid is projected onto a line, using an angle α.

3) The 'original' basis function b_i associated with box B_i is defined as

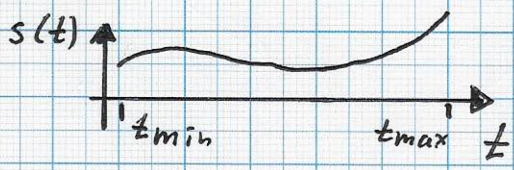
$$b_i(x) = \begin{cases} 1, & x \in B_i \\ 0, & \text{otherwise} \end{cases}$$

Towards Optimal Reconstruction... - Cont'd.

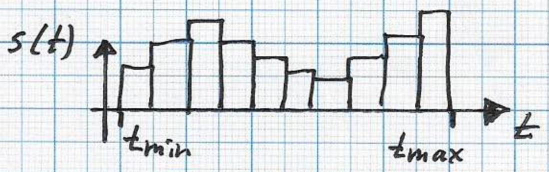
4) The unknown bivariate piecewise-constant function is $\underline{B(*)} = \sum_i c_i \cdot b_i(*)$.

(The coefficients c_i are unknown.)

5) Given/known/measured: a univariate $s(t)$ SINOGRAM function over the t-line:



smooth sinogram



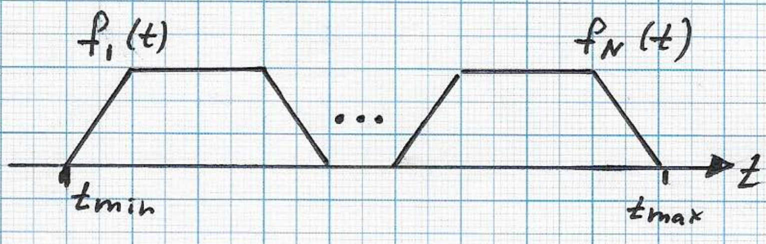
discrete sinogram

PROJECTION of LINEAR COMBINATION of BOX BASIS FUNCTIONS
≡ LINEAR COMBINATION of PROJECTIONS of BOX BASIS FUNCTIONS

6) *** METHOD ***

i) Compute a BEST APPROXIMATION of the sinogram $s(t)$ using the known (box projection) basis functions $f_i(t)$:

$$\boxed{app(t) = \sum_i c_i\, f_i(t)}$$



N = total number of boxes/pixels

ii) Knowing the c_i-values, compute $\underline{B(*)}$.

iii) Functions $\underline{f_i(t)}$ must be LINEARLY INDEPENDENT!

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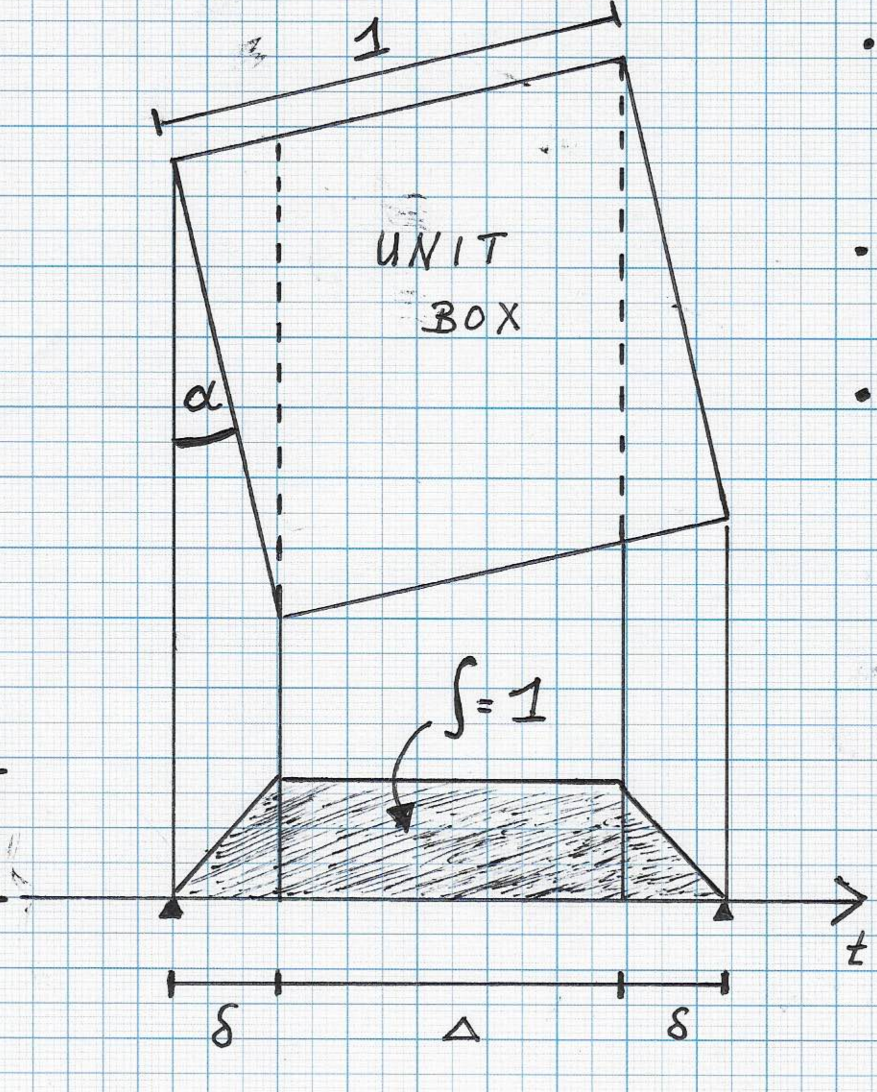
■ Towards Optimal Reconstruction – cont'd.

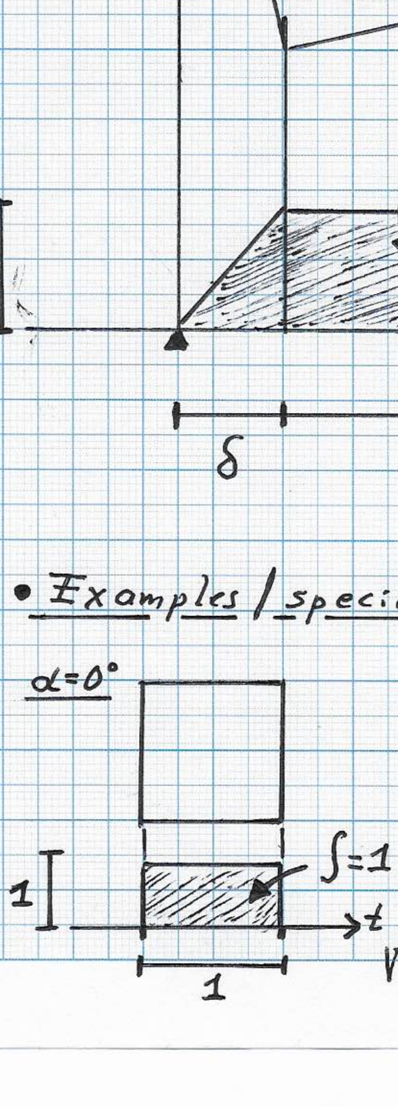
– The Basis Functions Obtained via Projection

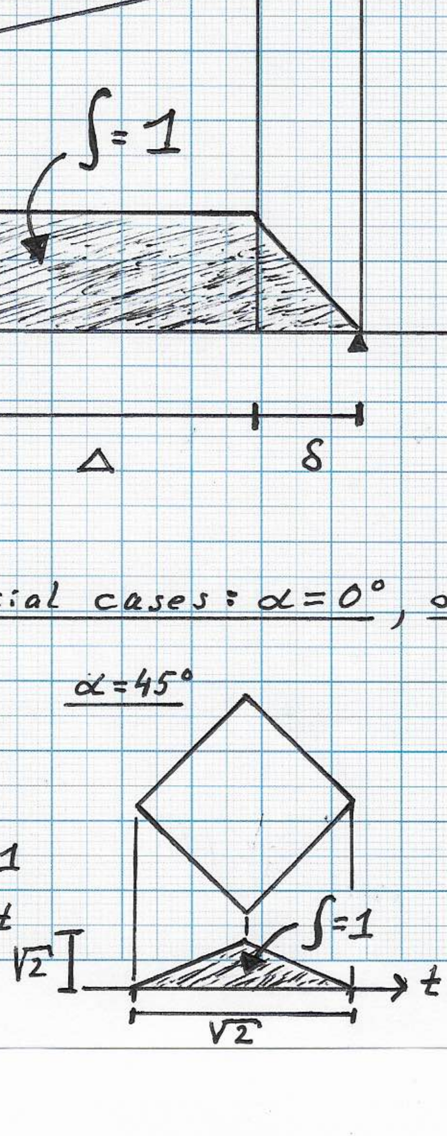


- Projection of UNIT BOX function onto line
- Projection defined by angle α
- Resulting three parameters defining the basis function on the line:

$$\boxed{\begin{array}{l} \delta = \sin(\alpha) \\ \quad (\cos(\alpha) = \Delta + \delta) \\ \Delta = \cos(\alpha) - \sin(\alpha) \\ \quad (\delta h + \Delta h = 1) \\ h = 1/\cos(\alpha) \end{array}}$$

- Examples / special cases: $\alpha = 0°$, $\alpha = 45°$:





(i) $\alpha = 0°$:

$\delta = 0$, $\Delta = 1$, $h = 1$

(ii) $\alpha = 45°$:

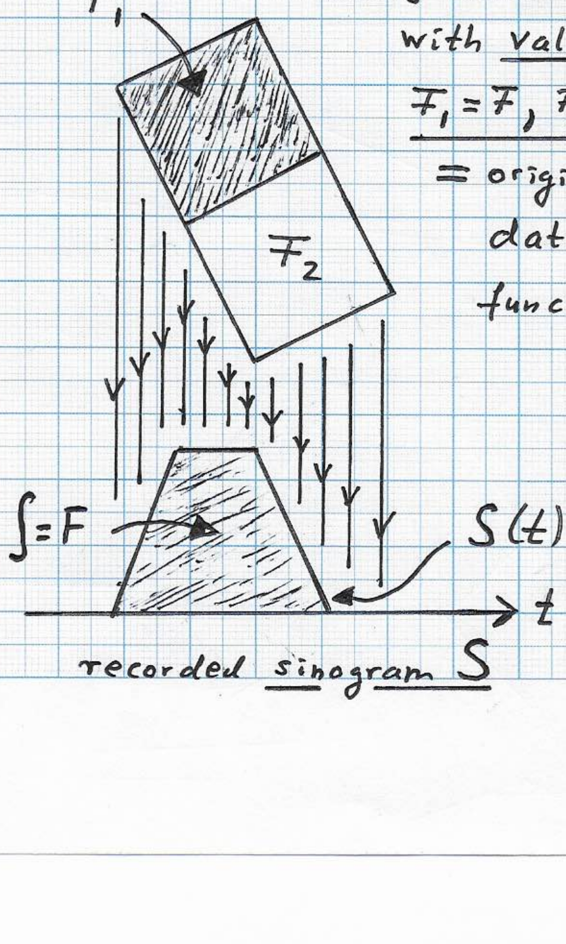
$\delta = \sqrt{2}/2$, $\Delta = 0$, $h = \sqrt{2}$

Towards Optimal Reconstruction – cont'd.

– Examples & Special Considerations

→ **PRINCIPLE:** "Compute the best linear combination of the univariate projections of the BOX functions (defined on a fixed Cartesian grid) to approximate (with minimal error) a given sinogram; use the same coefficients to linearly combine the bivariate BOX functions to obtain a reconstruction on the 2D Cartesian grid."

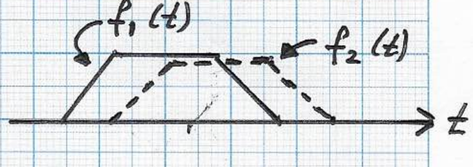
→ **Example:**



recorded sinogram S

- 2 grid cells with values $F_1 = F$, $F_2 = 0$ = original data/function

- corresponding projected basis functions for 2 cells:



⇒ Best approximation of S: $S(t) = F \cdot f_1(t) + 0 \cdot f_2(t)$

⇒ Reconstruction:

$F \cdot Box_1(x,y) + 0 \cdot Box_2(x,y)$,

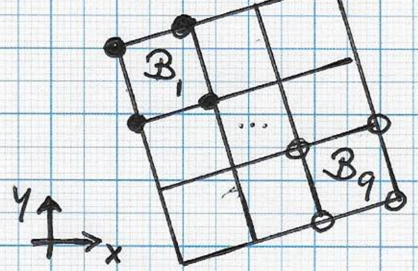
where

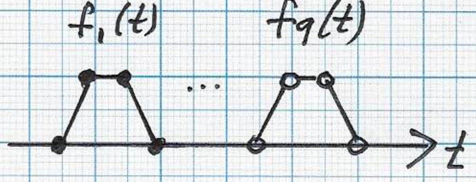
$$Box_i(x,y) = \begin{cases} 1 & \text{if } (x,y) \in BOX_i \\ 0 & \text{otherwise} \end{cases}$$

■ Towards Optimal Reconstruction – cont'd.

– The General Setting & BEST APPROXIMATION

i) Generic Cartesian Grid and 'Projected' Basis Functions:

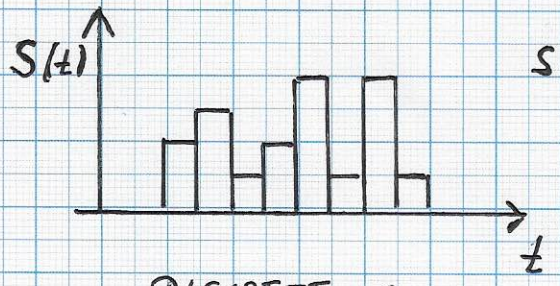


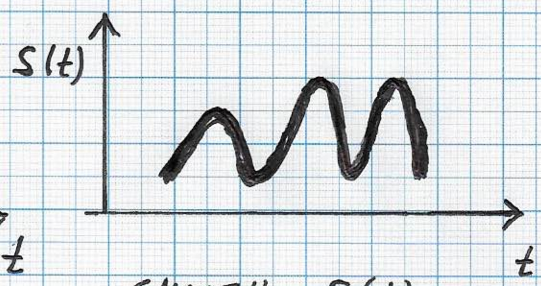


Cartesian grid
⇒ Box basis functions
$B_1, \ldots, B_q$

Corresponding projected basis functions
$f_1, \ldots, f_q$
(ordered "left-to-right")

ii) The Sinogram – Approximated Optimally with Fcts. $f_i(t)$:





DISCRETE, piecewise constant $S(t)$ – e.g., 1024 values

SMOOTH $S(t)$ ("∞ resolution)

iii) Minimize APPROXIMATION ERROR $E = \int \left(S(t) - \sum c_i f_i(t) \right)^2$

by solving the NORMAL EQUATIONS:

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▪ Towards Optimal Reconstruction – cont'd.

– Normal Equations to Be Solved

Normal Equations:

$$\begin{bmatrix} \langle f_1(t), f_1(t)\rangle & \dots & \langle f_1(t), f_N(t)\rangle \\ \vdots & & \vdots \\ \langle f_N(t), f_1(t)\rangle & \dots & \langle f_N(t), f_N(t)\rangle \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ \vdots \\ c_N \end{bmatrix} = \begin{bmatrix} \langle S(t), f_1(t)\rangle \\ \vdots \\ \langle S(t), f_N(t)\rangle \end{bmatrix}$$

Notes:

- $$\left.\begin{aligned} \langle f_i(t), f_j(t)\rangle &= \int f_i(t)\cdot f_j(t)\, dt \\ \langle S(t), f_i(t)\rangle &= \int S(t)\cdot f_i(t)\, dt \end{aligned}\right\}$$ inner/scalar products

  $S(t)$ is smooth ⇒ numerical integration

- Inner products $\langle f_i(t), f_j(t)\rangle$ can be computed directly, based on general symbolic integration of two functions $f_i(t)$ and $f_j(t)$.

- ONLY SOME INNER PRODUCTS $\langle f_i(t), f_j(t)\rangle$ ARE $\neq 0$ SINCE THE DOMAIN WHERE A FUNCTION $f_i(t) \neq 0$ IS FINITE.

  ⇒ THE SYSTEM MATRIX IS BANDED !!!

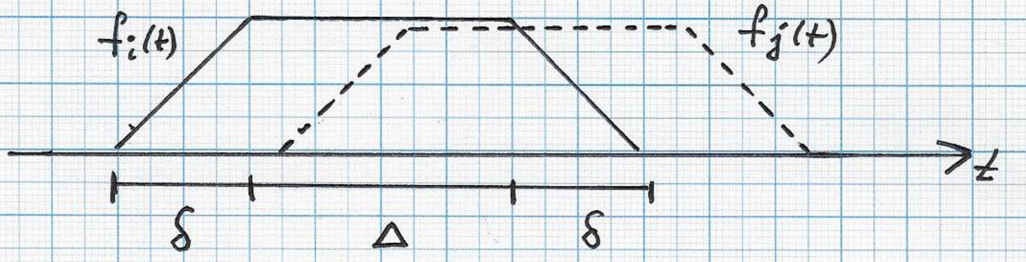
- ⇒ THE NORMAL EQUATIONS CAN BE SOLVED IN LINEAR TIME!!!

[Must ensure that functions $f_i(t)$ are ordered from 'left to right' on the t-axis.]

Towards Optimal Reconstruction – cont'd.

Notes (cont'd.):

- Integrate $\langle f_i, f_j \rangle$ symbolically for all functions $f_i(t)$ and $f_j(t)$ whose non-zero domains overlap:



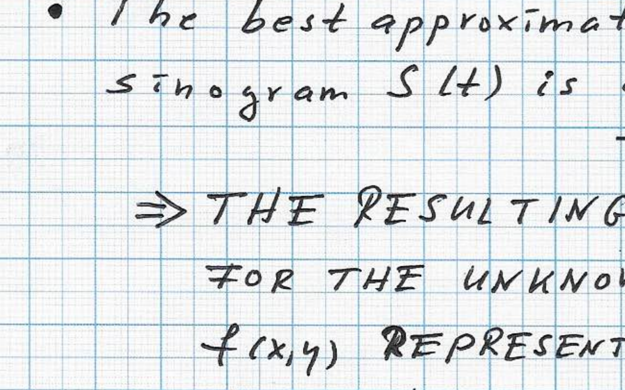
$\Rightarrow \int f_i(t) \cdot f_j(t) \, dt =$ EXACT SYMBOLIC FORMULA

(e.g., use "MATHICS".)

- The best approximation app (t) of a sinogram $S(t)$ is $\text{app}(t) = \sum_{i=1}^{N} c_i f_i(t)$.

$\Rightarrow$ THE RESULTING OPTIMAL RECONSTRUCTION FOR THE UNKNOWN BIVARIATE FUNCTION $f(x,y)$ REPRESENTED BY N CORRESPONDING BOXES / BOX BASIS FUNCTIONS $b_i(x,y) = b_i(\mathbf{x})$ IS

$$B(\mathbf{x}) = \sum_{i=1}^{N} c_i \cdot b_i(\mathbf{x}).$$



$\Leftarrow$ Ex: $N = 16$

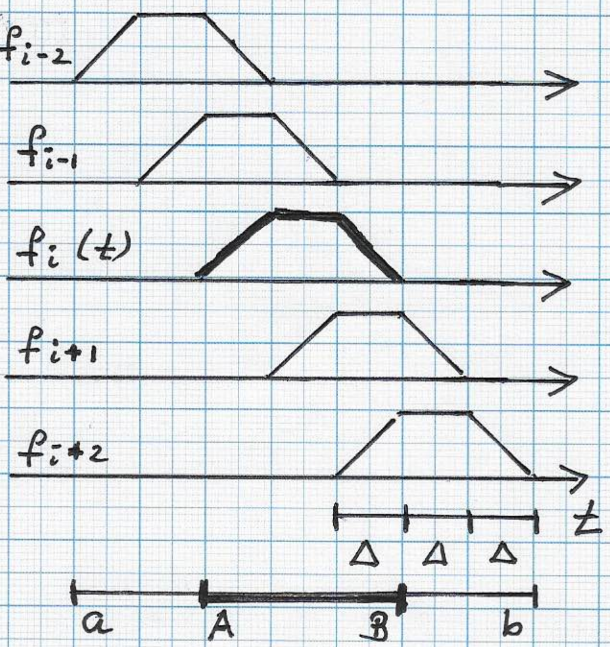
- discrete sinogram $S(t)$

— best approximation app (t) (constant and linear parts)

▪ Towards Optimal Reconstruction – cont'd.

iv) Special type of basis function $f_i(t)$:

$\underline{\delta = \Delta}$ (good for numerical reasons ?)



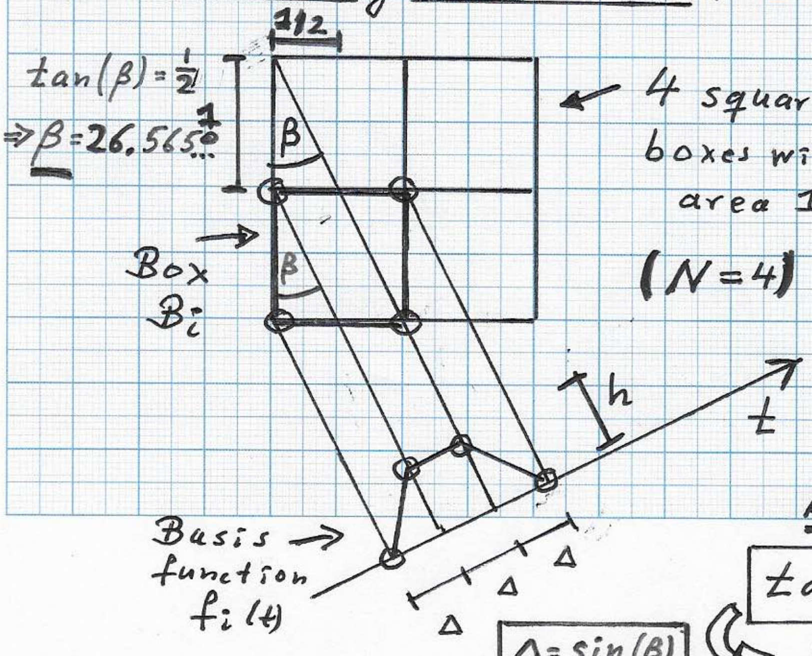
Here: (only 5 basis fcts $\neq 0$ in $[A,B]$)

Basis function $f_i(t)$ has non-zero inner products only with five other basis functions: $f_{i-2}, f_{i-1}, f_i, f_{i+1}, f_{i+2}$

$\Rightarrow$ **THE NORMAL EQUATIONS MATRIX HAS FIVE BANDS!**

(for this special case...)

→ Necessary Construction: (for $\delta = \Delta$)



4 square boxes with area 1

$(N=4)$

Condition:

$$\int f_i(t)\,dt = 1$$

$$= \left(\frac{\Delta}{2} + \Delta + \frac{\Delta}{2}\right) \cdot h$$

$$= 2\Delta h$$

$$\Rightarrow \boxed{h = 1/2\Delta}$$

AND

$\boxed{\tan(\beta) = 1/n}$, where $N = n^2$ (here $n = 2$)

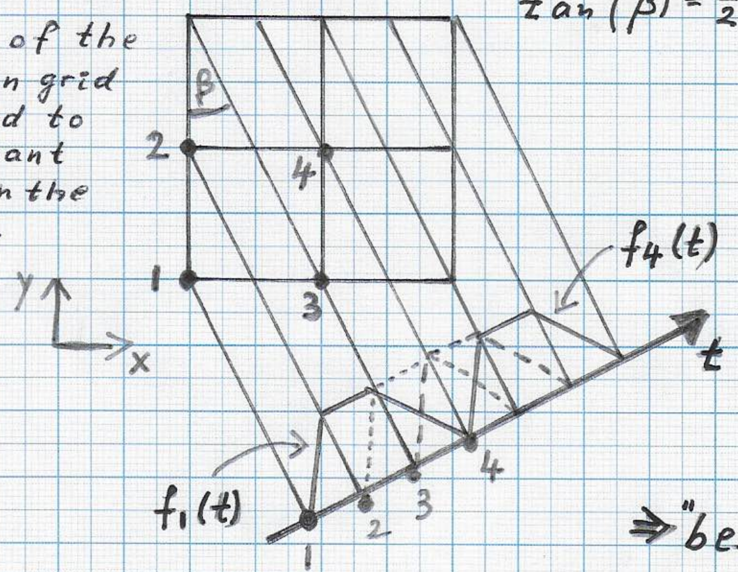
$\boxed{\Delta = \sin(\beta)}$ $\Rightarrow$ $\underline{\delta = \Delta}$ $\boxed{= 1/2}$

▪ Towards Optimal Reconstruction - cont'd.

iv) Cont'd: Special examplary case of 2×2 grid:

! GOAL: Vertices of the Cartesian grid projected to equidistant points on the t-line.

$\tan(\beta) = \frac{1}{2}$



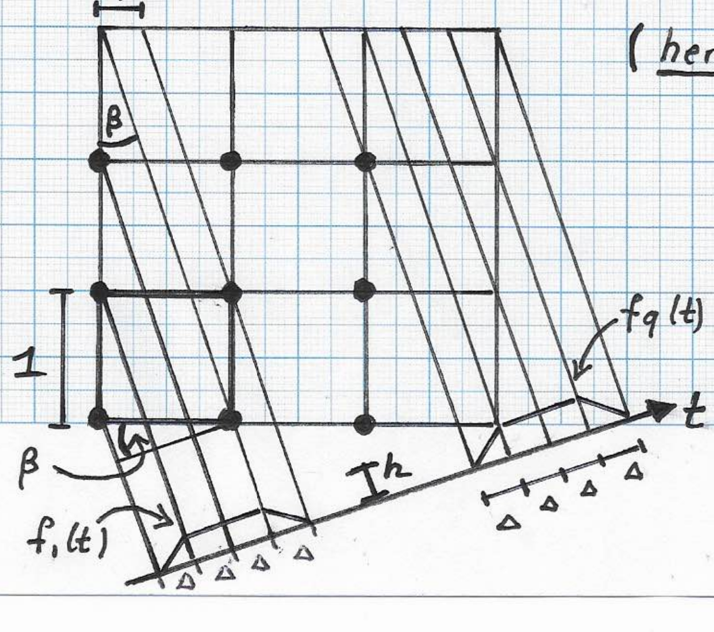
- Lower left corner of a box (with respect to x, y)

→ boxes sorted from left to right using • point and t-axis

⇒ "best-possible sorting" of linearly independent basis functions $f_i(t)$, $i = 1, 2, 3, 4$

OPTIMAL RECONSTRUCTION POSSIBLE FOR THIS SPECIAL CASE!

v) The General Case: $n > 2$:



(here: $n = 3$)

- $\tan(\beta) = \frac{1}{n} \Rightarrow \beta = \ldots$
- sorted basis functions $f_1, \ldots, f_{n^2}$
- $\cos(\beta) = n \cdot \Delta$

$\Rightarrow \underline{\underline{\Delta = \frac{1}{n} \cdot \cos(\beta)}}$

- $\int f_i(t)\, dt = 1 = n \cdot \Delta \cdot h$

$\Rightarrow \underline{\underline{h = \frac{1}{n \cdot \Delta}}}$

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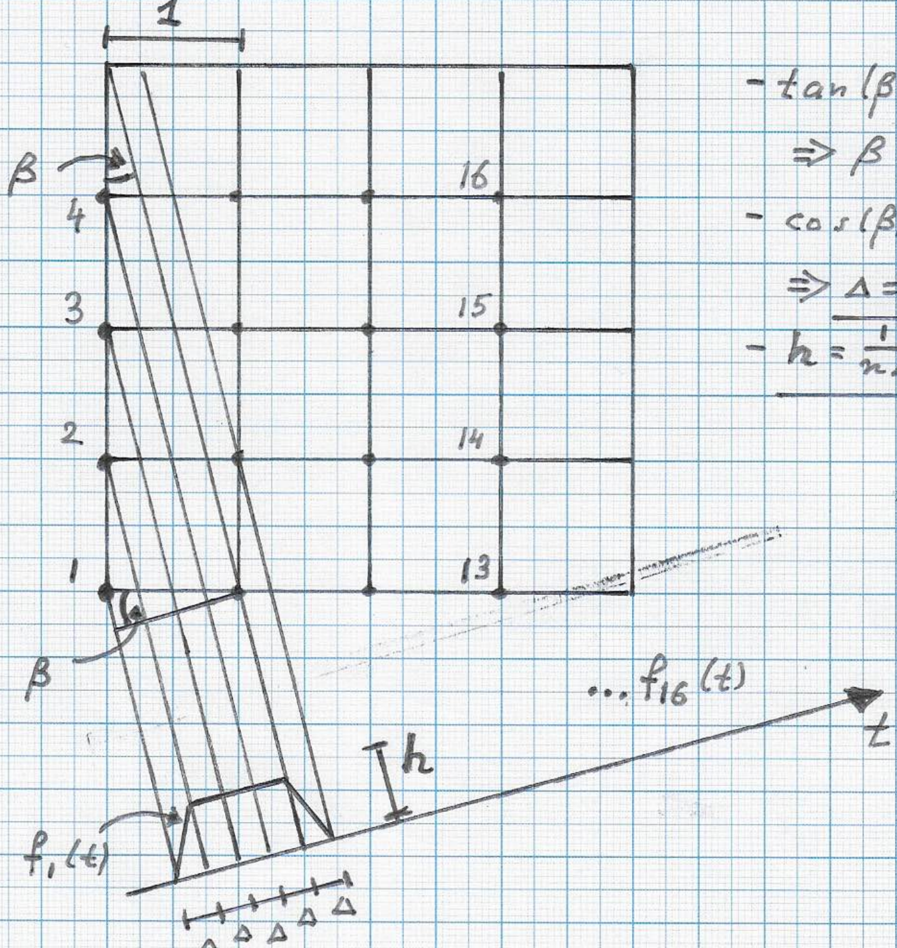
■ Towards Optimal Reconstruction – cont'd.

v) Cont'd: Example with $n = 4$



- $\tan(\beta) = \frac{1}{n} = \frac{1}{4}$

  $\Rightarrow \beta = \ldots$

- $\cos(\beta) = n \cdot \Delta = 4\Delta$

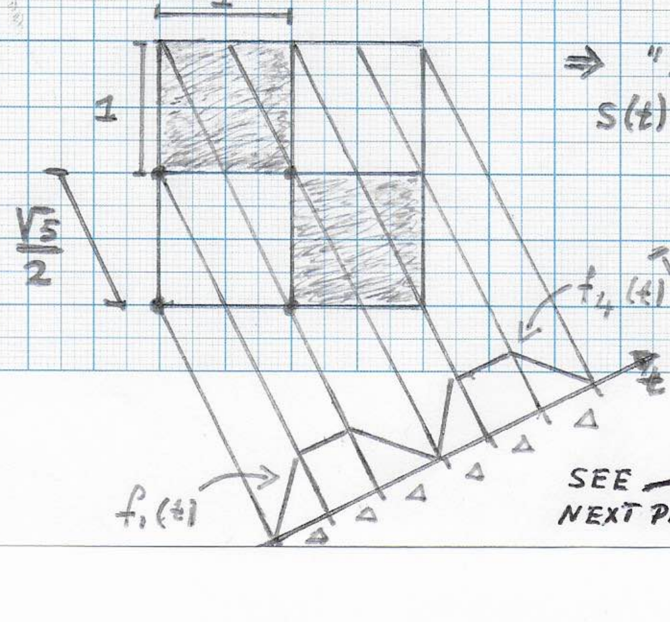
  $\Rightarrow \Delta = \frac{1}{4} \cos(\beta)$

- $h = \frac{1}{n\Delta} = \frac{1}{4\Delta}$

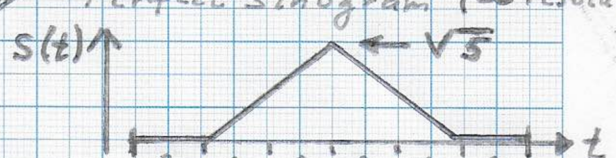
vi) Numerical example for $n = 2$:

to be reconstructed:

 $= \begin{array}{|c|c|} \hline 1 & 0 \\ \hline 0 & 1 \\ \hline \end{array}$



$\Rightarrow$ "Perfect sinogram" (∞ resolution):



$\Rightarrow$ Perfect reconstruction via NORMAL EQUATIONS:

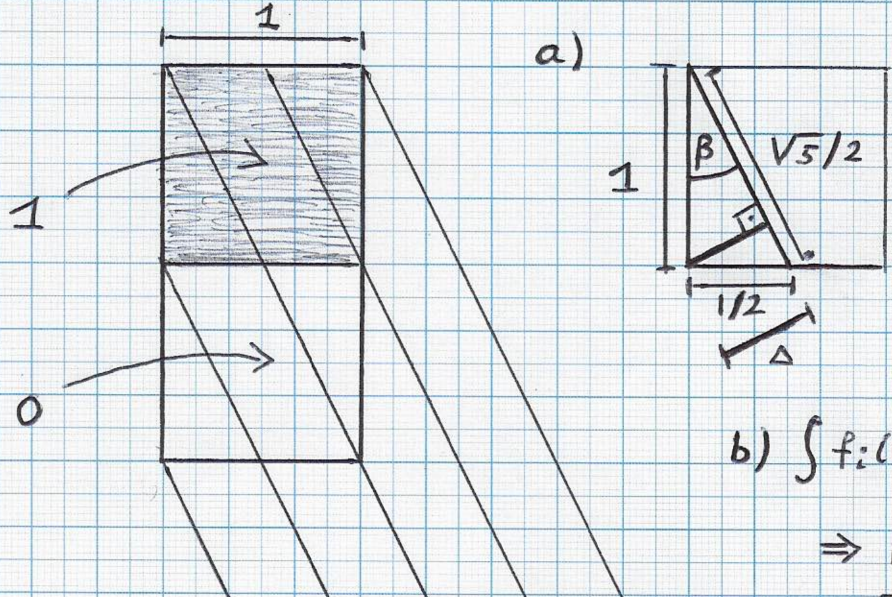
$$s(t) = 0 \cdot f_1 + \frac{\sqrt{5}}{2h} f_2 + \frac{\sqrt{5}}{2h} f_3 + 0 f_4$$

SEE NEXT PAGE $= \mathbf{0} \cdot f_1 + \mathbf{1} \cdot f_2 + \mathbf{1} \cdot f_3 + \mathbf{0} \cdot f_4$

Towards Optimal Reconstruction – cont'd.

vi) Numerical examples – cont'd.

→ "Method reproduces a density function EXACTLY when the original density function is a piecewise-constant function on a Cartesian grid:



→ $\sin(\beta) = \Delta$

→ $\sin(\beta) = \frac{1}{2} : \frac{\sqrt{5}}{2} = \sqrt{5}/5$

$\Rightarrow \underline{\underline{\Delta = \sqrt{5}/5}}$

b) $\int f_i(t)\,dt \overset{!}{=} 1 = 2\Delta h$

$\Rightarrow \underline{\underline{h = \frac{1}{2\Delta} = \frac{\sqrt{5}}{2}}}$

c) System:

$\underline{\underline{s(t) = \sum_{i=1}^{2} c_i f_i(t)}}$

→ $c_1 f_1 = c_1 h \overset{!}{=} 0$

$\Rightarrow \underline{c_1 = 0}$

→ $c_2 f_2 = c_2 h \overset{!}{=} \sqrt{5}/2$

$\Rightarrow \underline{\underline{c_2 = \frac{1}{h} \cdot \frac{\sqrt{5}}{2} = \frac{2}{\sqrt{5}} \cdot \frac{\sqrt{5}}{2} = 1}}$

METHOD RECONSTRUCTS EXACTLY WHAT IT SHOULD REPRODUCE!

☺ BH

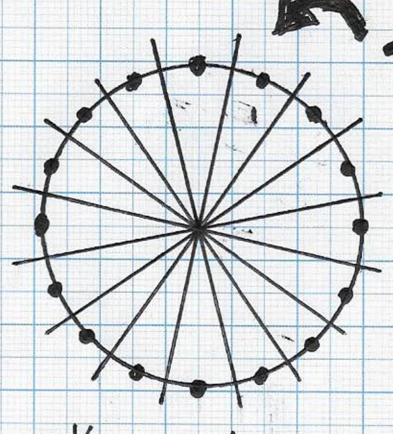
■ Towards Optimal Reconstruction - cont'd.

NOTES:

→ Sinogram is discrete, has finite resolution.

⇒ Compute reconstructions for several sinograms and COMBINE INDIVIDUAL RECONSTRUCTIONS!

Example:
- 1 sinogram = 1024 values
- total number of sinograms = 720
- for each sinogram compute a **32×32** (Cartesian mesh-based), generating 1024 pixel intensity values (per sinogram)
- COMBINE ALL 720 × 32 × 32 RECONSTRUCTED VALUES by interpreting the 720 × 32 × 32 values as values at UNCONNECTED POINTS (= centers of pixels). The unconnected points can be viewed as sites of a VORONOI DIAGRAM, which can be used as FINAL RECONSTRUCTION!



= Voronoi diagram of 4 sinograms of 2×2

→ Hierarchical, MULTIRESOLUTION RECONSTRUCTION: Use as basis functions for sinogram approximation the projections of the BIVARIATE HAAR WAVELET FUNCTIONS defined over the **2D** region used for the bivariate reconstruction. ?

BH