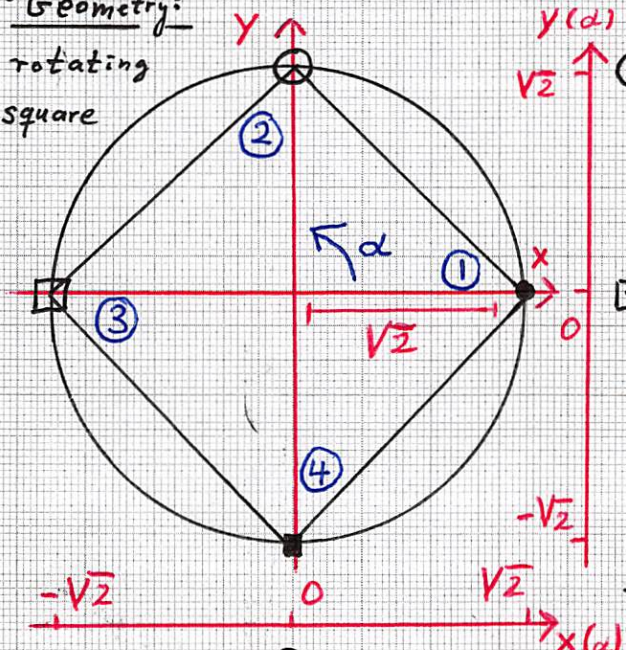
Bernd Hamann

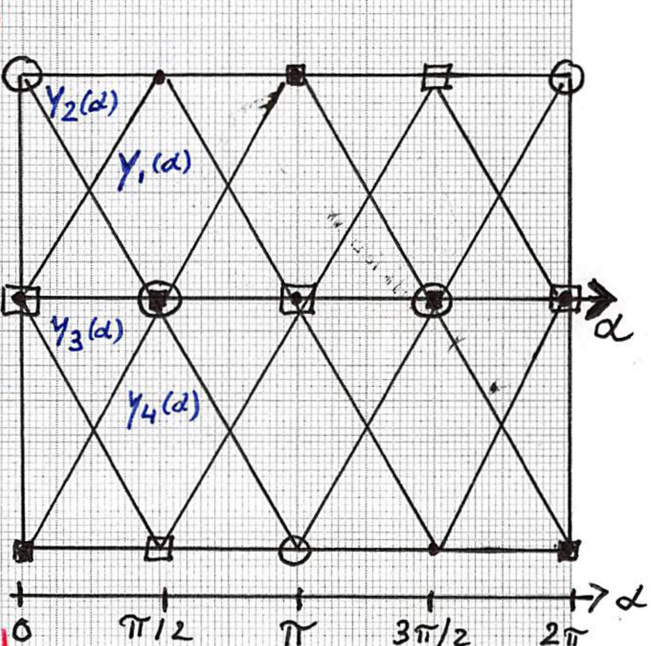
12/9/2019

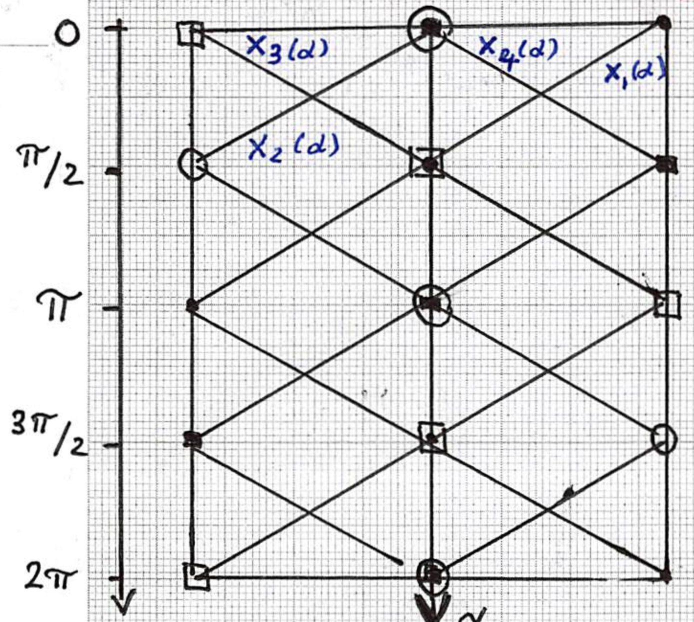
Stratovan

RECONSTRUCTION: PROJECTION OF A UNIT SQUARE, ANALYTICAL SOLUTION

• **Geometry:** rotating square







$y_1(\alpha) = \sin\alpha$

$y_2(\alpha) = \cos\alpha$

$y_3(\alpha) = -\sin\alpha$

$y_4(\alpha) = -\cos\alpha$

$x_1(\alpha) = \cos\alpha$

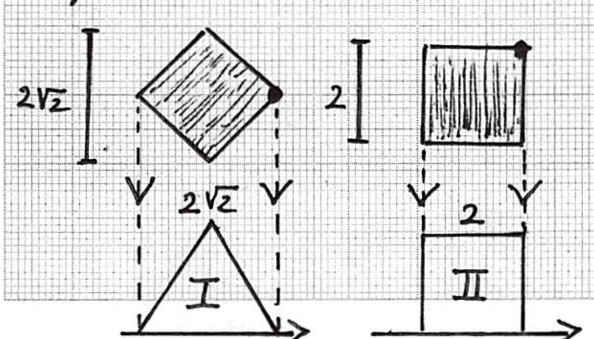
$x_2(\alpha) = -\sin\alpha$

$x_3(\alpha) = -\cos\alpha$

$x_4(\alpha) = \sin\alpha$

$\Rightarrow$ $\begin{pmatrix} x_i(\alpha) \\ y_i(\alpha) \end{pmatrix} =$ "movement" of point ⓘ on circle with radius $\sqrt{2}$; $\alpha \in [0, 2\pi]$

• **GOAL**: ANALYTICALLY DEFINED PROJECTIONS!



Projections

$\Leftarrow$ Two extreme projection cases:

I) projection = hat function

II) projection = box function

$\Rightarrow$ The rotating square produces projections with 2 linear pieces and 1 constant piece.

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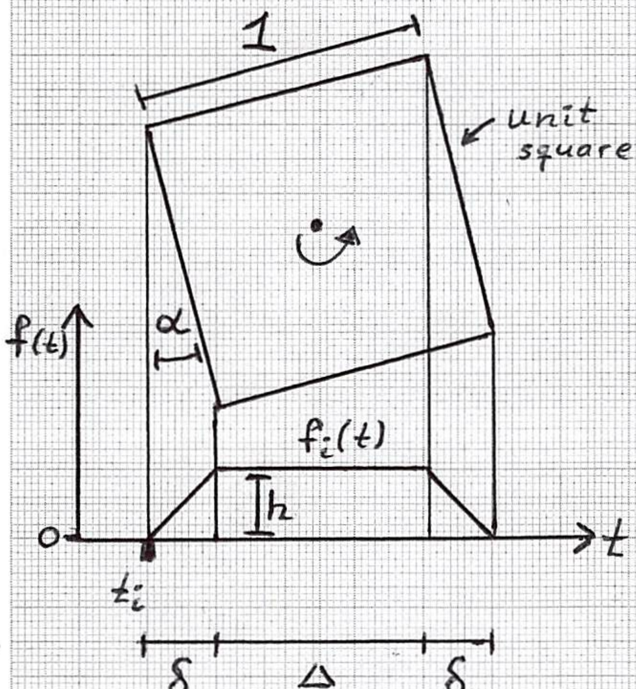
■ RECONSTRUCTION: SQUARE PROJECTION, ANALYTICAL SOLUTION

(➨ See notes from 4/2/2018, pp. 3ff)



- Rotating square produces smoothly varying projections — BLaC functions $f_i(t)$, depending on angle α

- Consider case $0 \le \alpha \le \pi/4$:

$$\boxed{\begin{aligned} \delta &= \sin\alpha = s\alpha \\ \Delta &= \cos\alpha - \sin\alpha = c\alpha - s\alpha \\ h &= 1/(\delta+\Delta) \end{aligned}}$$

- $$f_i(t) = \begin{cases} 0, & t < t_i \\ \frac{h}{\delta}(t - t_i), & t_i \le t < t_i + \delta \\ \ldots \\ 0, & t > t_i + 2\delta + \Delta \end{cases}$$

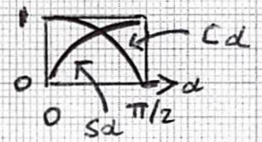
$\Rightarrow f_i(t)$ = function with constant & linear segments

$\Rightarrow \langle f_i(t), f_j(t) \rangle = \ldots$ **ANALYTIC DEFINITION!**

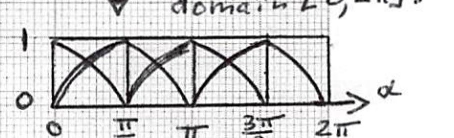
- Full rotation, i.e., $\alpha \in [0, 2\pi]$:

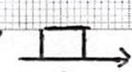
α*	δ	Δ
$0 \ldots \frac{\pi}{4}$	$s\alpha$	$c\alpha - s\alpha$
$\frac{\pi}{4} \ldots \frac{\pi}{2}$	$c\alpha$	$s\alpha - c\alpha$
$\frac{\pi}{2} \ldots \frac{3\pi}{4}$	$s(\alpha - \frac{\pi}{2})$	$c(\alpha - \frac{\pi}{2}) - s(\alpha - \frac{\pi}{2})$
$\frac{3\pi}{4} \ldots \pi$	$c(\alpha - \frac{\pi}{2})$	$s(\alpha - \frac{\pi}{2}) - c(\alpha - \frac{\pi}{2})$
$\pi \ldots \frac{5\pi}{4}$	$s(\alpha - \pi)$	$c(\alpha - \pi) - s(\alpha - \pi)$
$\frac{5\pi}{4} \ldots \frac{3\pi}{2}$	$c(\alpha - \pi)$	$s(\alpha - \pi) - c(\alpha - \pi)$
$\frac{3\pi}{2} \ldots \frac{7\pi}{4}$	$s(\alpha - \frac{3\pi}{2})$	$c(\alpha - \frac{3\pi}{2}) - s(\alpha - \frac{3\pi}{2})$
$\frac{7\pi}{4} \ldots 2\pi$	$c(\alpha - \frac{3\pi}{2})$	$s(\alpha - \frac{3\pi}{2}) - c(\alpha - \frac{3\pi}{2})$

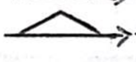
* Only consider $0 \le \alpha \le \pi$ practically.

- 

this pattern 4 times repeated for domain $[0, 2\pi]$:



-  $t \iff \alpha = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$

 $t \iff \alpha = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

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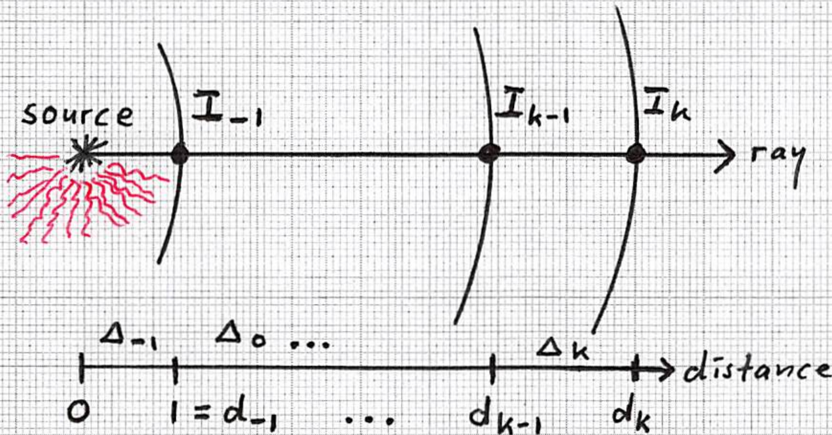
■ POINT LIGHT SOURCE CT DATA AND RECONSTRUCTION

– Combining Inverse Distance Law and Absorption

I) Inverse Distance Law

Law: Given a "point light source," photon density decreases proportionally to distance (2D) or $(\text{distance})^2$ (3D) to the light source location.



I_{-1} = intensity at distance 1

I_{k-1} = intensity at distance d_{k-1}

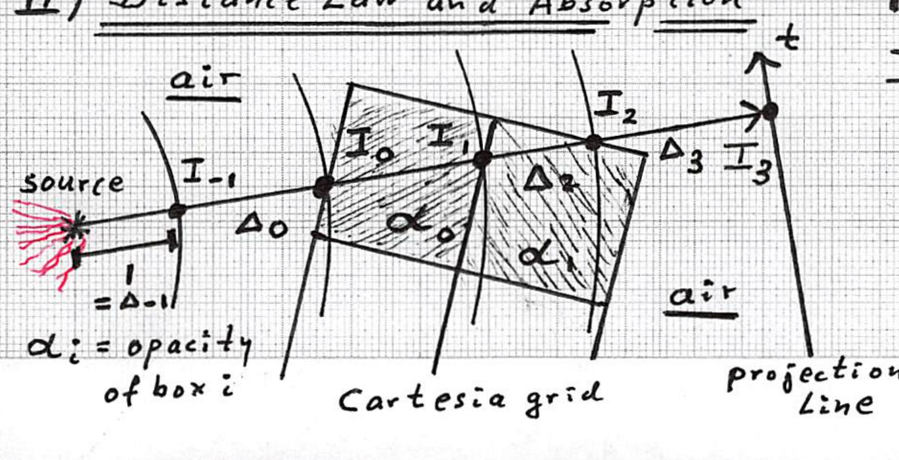
etc.

$$\Rightarrow I_{k-1} = \left(\frac{1}{d_{k-1}}\right)^p I_{-1} \curvearrowright I_{-1} = (d_{k-1})^p I_{k-1} \quad (*)$$

$$I_k = \frac{1}{(d_k)^p} I_{-1} \overset{(*)}{=} \left(\frac{d_{k-1}}{d_k}\right)^p I_{-1}$$

$$p = \begin{cases} 1 & \rightarrow \text{2D case} \\ 2 & \rightarrow \text{3D case} \end{cases}$$

II) Distance Law and Absorption



Here:

$$I_0 = \left(\frac{d_{-1}}{d_0}\right)^p I_{-1}$$

$$I_1 = \left(\frac{d_0}{d_1}\right)^p I_0 \cdot e^{-\Delta_0 \alpha_0}$$

$$= \left(\frac{d_{-1}}{d_1}\right)^p I_{-1} e^{-\Delta_0 \alpha_0}$$

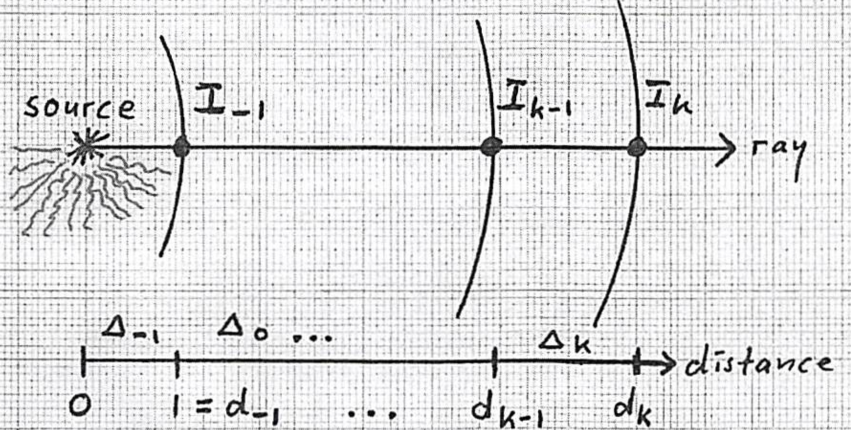
CORRECTION (2-26-2020)

POINT LIGHT SOURCE CT DATA AND RECONSTRUCTION

– Combining Inverse Distance Law and Absorption

I) Inverse Distance Law

Law: Given a "point light source," photon density decreases proportionally to distance (2D) or (distance)^2 (3D) to the light source location.



I_{-1} = intensity at distance 1

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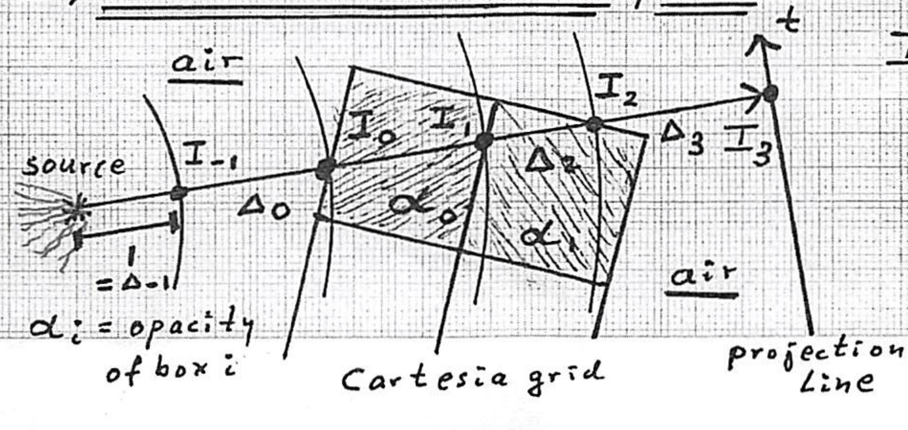
etc.

$$\Rightarrow I_{k-1} = \left(\frac{1}{d_{k-1}}\right)^p I_{-1} \quad \rightsquigarrow \quad I_{-1} = (d_{k-1})^p I_{k-1} \qquad (*)$$

$$I_k = \frac{1}{(d_k)^p} I_{-1} \overset{(*)}{=} \left(\frac{d_{k-1}}{d_k}\right)^p \cancel{I_{-1}}\; I_{k-1}$$

$$p = \begin{cases} 1 & \to 2D \text{ case} \\ 2 & \to 3D \text{ case} \end{cases}$$

II) Distance Law and Absorption



Here:

$$I_0 = \left(\frac{d_{-1}}{d_0}\right)^p I_{-1}$$

$$I_1 = \left(\frac{d_0}{d_1}\right)^p I_0 \cdot e^{-\Delta_{\cancel{0}1}\alpha_0}$$

$$= \left(\frac{d_{-1}}{d_1}\right)^p I_{-1}\, e^{-\Delta_{\cancel{0}1}\alpha_0}$$

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■ POINT LIGHT SOURCE – Cont'd.

... Absorption:

$$I_2 = \left(\frac{d_1}{d_2}\right)^p I_1 e^{-\Delta_1 \alpha_1}$$

$$= \left(\frac{d_{-1}}{d_2}\right)^p I_{-1} e^{-(\Delta_0 \alpha_0 + \Delta_1 \alpha_1)}$$

$$= \frac{1}{(d_2)^p} I_{-1} e^{-(\Delta_0 \alpha_0 + \Delta_1 \alpha_1)}$$

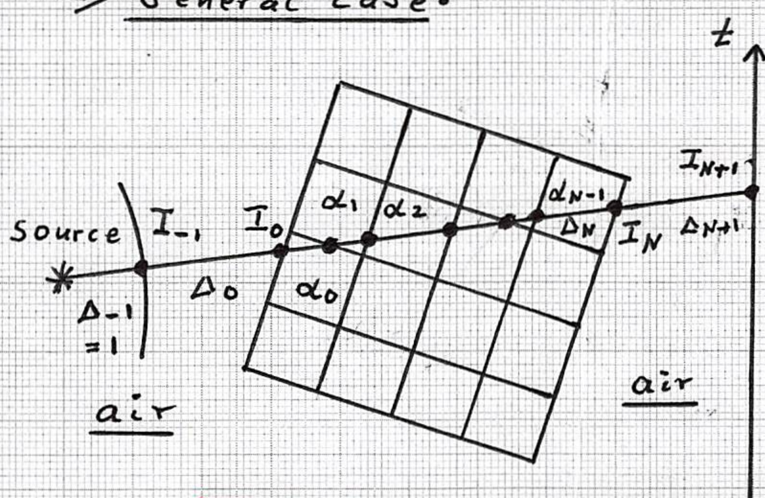
$$I_3 = \left(\frac{d_2}{d_3}\right)^p I_2$$

$$= \left(\frac{d_{-1}}{d_3}\right)^p I_{-1} e^{-(\Delta_0 \alpha_0 + \Delta_1 \alpha_1)}$$

$$= \frac{1}{(d_3)^p} I_{-1} e^{-(\Delta_0 \alpha_0 + \Delta_1 \alpha_1)}$$

$\Rightarrow$ LAW: $\boxed{\Delta_0 \alpha_0 + \Delta_1 \alpha_1 = -\ln\left(d_3^p \frac{I_3}{I_{-1}}\right)}$

$\Rightarrow$ General case:



(*) Equation defines linear system for unknown values α_i (opacities).

(*)

$$\boxed{\sum_{i=0}^{N-1} \Delta_i \alpha_i = -\ln\left(d_{N+1}^p \frac{I_{N+1}}{I_{-1}}\right)}$$

$p=1$: 2D

$p=2$: 3D

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CORRECTION (2-26-2020)

■ POINT LIGHT SOURCE – Cont'd.

... Absorption:

$$I_2 = \left(\frac{d_1}{d_2}\right)^p I_1 \, e^{-\Delta_{\cancel{x}\,2}\,\alpha_1}$$

$$= \left(\frac{d_{-1}}{d_2}\right)^p I_{-1} \, e^{-(\Delta_{\cancel{x}\,1}\,\alpha_0 + \Delta_{\cancel{x}\,2}\,\alpha_1)}$$

$$= \frac{1}{(d_2)^p} \, I_{-1} \, e^{-(\Delta_{\cancel{x}\,1}\,\alpha_0 + \Delta_{\cancel{x}\,2}\,\alpha_1)}$$

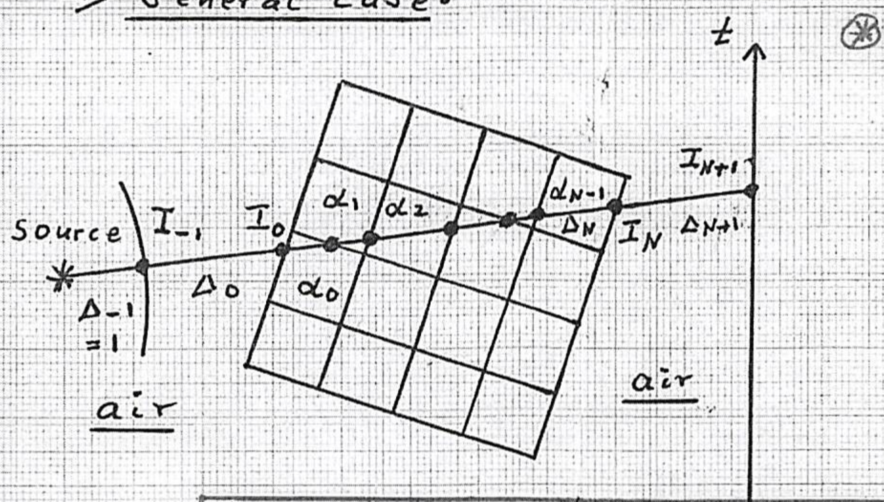
$$I_3 = \left(\frac{d_2}{d_3}\right)^p I_2$$

$$= \left(\frac{d_{-1}}{d_3}\right)^p I_{-1} \, e^{-(\Delta_{\cancel{x}\,1}\,\alpha_0 + \Delta_{\cancel{x}\,2}\,\alpha_1)}$$

$$= \frac{1}{(d_3)^p} \, I_{-1} \, e^{-(\Delta_{\cancel{x}\,1}\,\alpha_0 + \Delta_{\cancel{x}\,2}\,\alpha_1)}$$

$\Rightarrow$ LAW: $$\boxed{\Delta_{\cancel{x}\,1}\,\alpha_0 + \Delta_{\cancel{x}\,2}\,\alpha_1 = -\ln\left(d_3^p \frac{I_3}{I_{-1}}\right)}$$

$\Rightarrow$ General case:



⊛ Equation defines linear system for unknown values α_i (opacities).

⊛

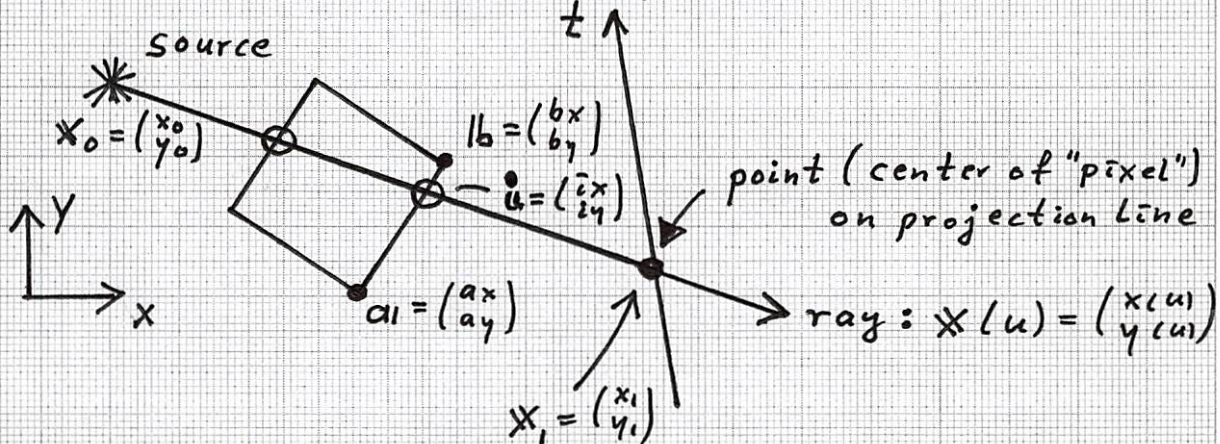
$$\boxed{\sum_{i=0}^{N-1} \Delta_{\cancel{x}\,i+1}\,\alpha_i = -\ln\left(d_{N+1}^{\,p} \frac{I_{N+1}}{I_{-1}}\right)}$$

$p=1$: 2D

$p=2$: 3D

POINT LIGHT SOURCE – Cont'd.

- Necessary computation (2D case):
  Intersections of ray and edges of grid!



i) Parametric ray equation:

(i) $$\underline{\mathbf{x}(u)} = \mathbf{x}_0 + u \underbrace{(\mathbf{x}_1 - \mathbf{x}_0)}_{= \mathbf{d} = \begin{pmatrix} d_x \\ d_y \end{pmatrix}} \underline{= \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + u \begin{pmatrix} d_x \\ d_y \end{pmatrix}}$$

ii) Implicit edge equation:

- edge direction vector: $\mathbf{e} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} e_x \\ e_y \end{pmatrix}$
- Line implied by edge defined implicitly:
- Implicit line representation:

(ii) $$Ax + By + C = 0, \quad \text{where } \underline{A = -e_y}, \ \underline{B = e_x}$$

$$\hookrightarrow \quad \underline{C} = -Ax - By \quad \Big| \text{ select } \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_x \\ a_y \end{pmatrix}$$
$$= -A a_x - B a_y$$
$$\underline{= a_x e_y - a_y e_x}$$

iii) Intersection: Insert (i) into (ii):

$$A(x_0 + \bar{u} d_x) + B(y_0 + \bar{u} d_y) = -C$$

$$\hookrightarrow \quad \underline{\underline{\bar{u} = -\frac{Ax_0 + By_0 + C}{Ad_x + Bd_y}}} \Rightarrow \underline{\underline{\mathbf{i} = \mathbf{x}_0 + \bar{u}\,\mathbf{d}}}$$

- $\underline{Ad_x + Bd_y \neq 0}$
- $\mathbf{i}$ inside $\mathbf{a}\mathbf{b}$ $\Leftrightarrow \underline{0 \leq \bar{u} \leq 1}$

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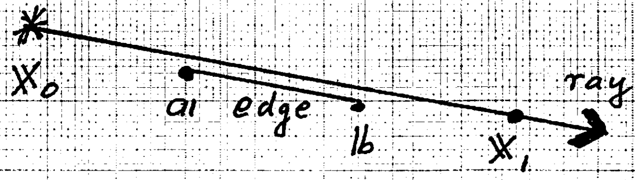
■ **POINT LIGHT SOURCE – Cont'd.**

... iii) Intersection – Special case:



$$A d_x + B d_y = 0$$

$\Rightarrow$ ray and edge are parallel!

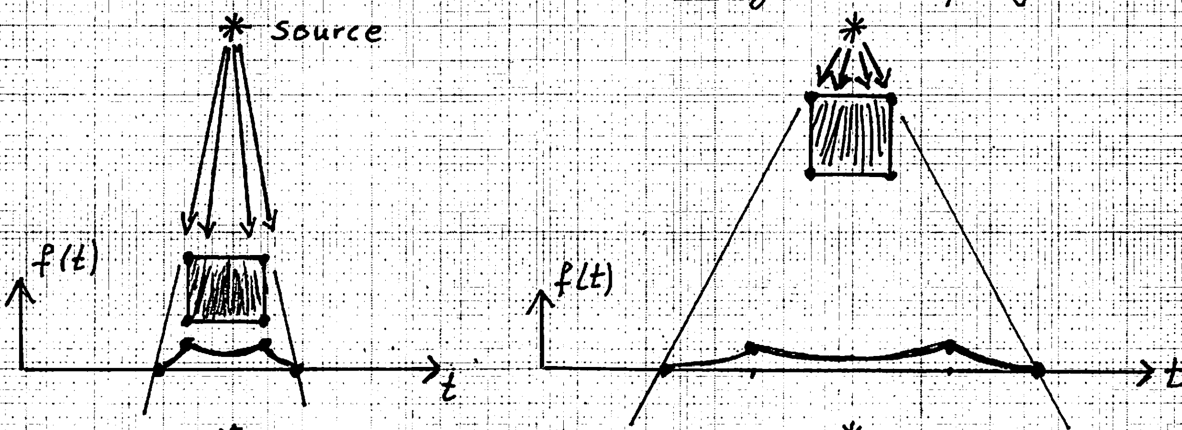
• 2 sub-cases:

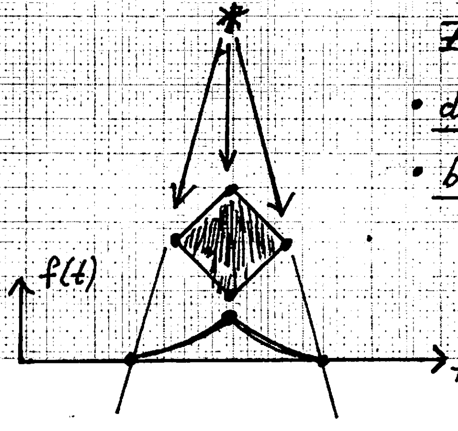
→ $A x_0 + B y_0 + C \neq 0 \Rightarrow$ no intersection!

→ $A x_0 + B y_0 + C = 0 \Rightarrow$ edge lies on ray

$\Rightarrow$ Select the point from $\{\mathbf{a}, \mathbf{b}\}$ that is closer to $\mathbf{x}_0$ as intersection point!

• "Qualitative nature of single box projections:

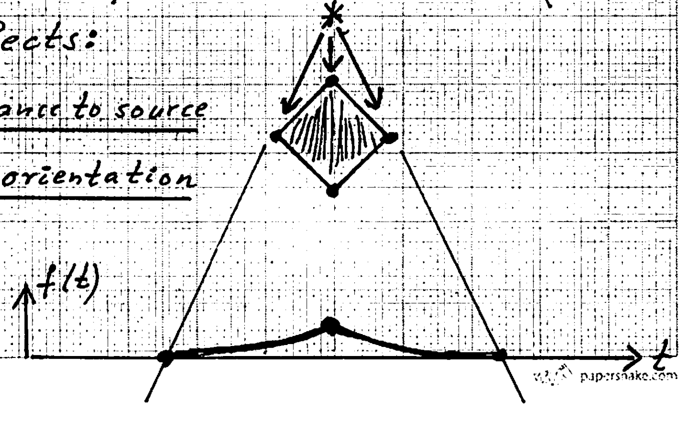




Effects:

• distance to source

• box orientation



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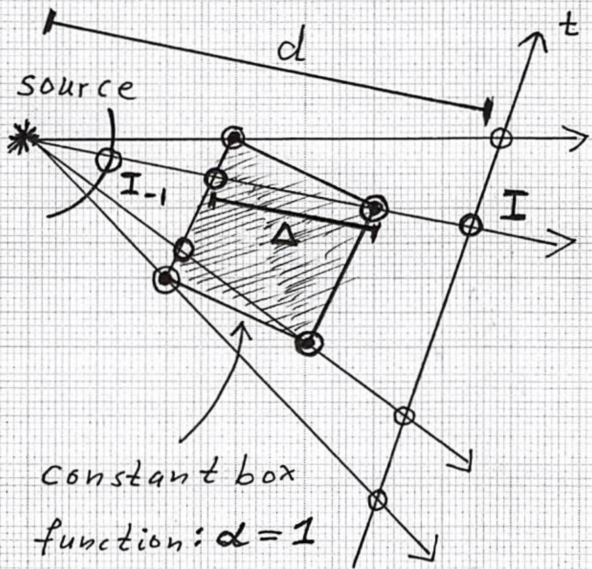
POINT LIGHT SOURCE - Cont'd.

- Projection of single box (function):



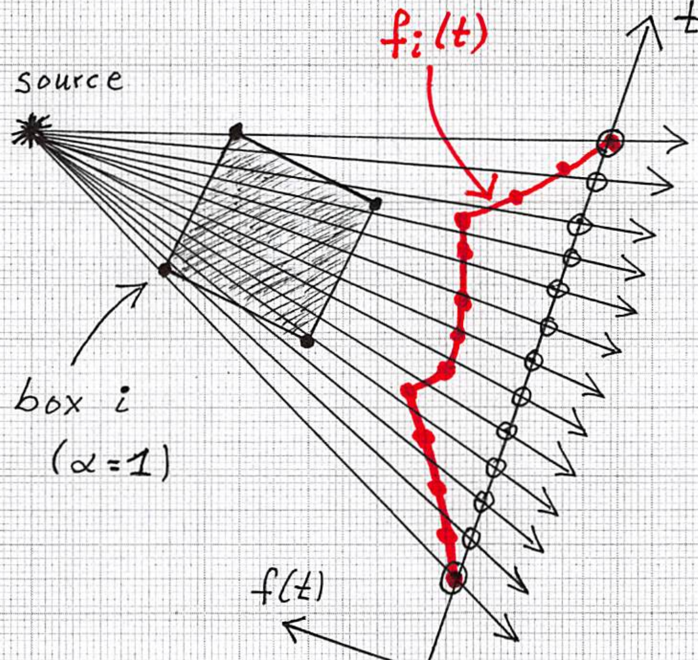
Inverse distance law and absorption:

$$\boxed{I = \frac{1}{d^p} I_{-1} e^{-\Delta}}$$

$p = 1 \Rightarrow 2D$
$p = 2 \Rightarrow 3D$

$\Rightarrow$ Use formula to comute projection functions of all box basis functions!

- Approximating projection of box function via sampling:



here:

o sample locations on projection line

• projection values I at locations o

$f_i(t)$ = approximation of projection function

$f_i(t)$ represented as piecewise constant or piecewise Linear function

POINT LIGHT SOURCE – Cont'd.

- Best approximation approach:

$f(t)$	given, measured CT scan data on projection line t
$F(x,y)$	unknown (opacity) function ("density function") in 2D plane
$B_i(x,y)$	constant box basis function, $i=1,\dots,N$ (opacity $=1$)
$f_i(t)$	computed, approximated projected box basis function
P	"projection operator"

unknown

$$\Rightarrow \underline{f(t)} = P\big(F(x,y)\big) = P\Big(\sum_{i=1}^{N} \alpha_i B_i(x,y)\Big)$$
$$= \sum_{i=1}^{N} \alpha_i P\big(B_i(x,y)\big)$$
$$= \underline{\sum_{i=1}^{N} \alpha_i f_i(t)}$$

$\Rightarrow$ Determine best-possible values for α_i via best approximation approach – CONSIDERING THAT BOTH $f(t)$ AND $f_i(t)$ ARE REPRESENTED IN DISCRETE, SAMPLED FORM:

- solution defined on high level:

$$\begin{pmatrix} \langle f_1, f_1\rangle & \dots & \langle f_1, f_N\rangle \\ \vdots & & \\ \langle f_N, f_1\rangle & \dots & \langle f_N, f_N\rangle \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{pmatrix} = \begin{pmatrix} \langle f, f_1\rangle \\ \vdots \\ \langle f, f_N\rangle \end{pmatrix}$$

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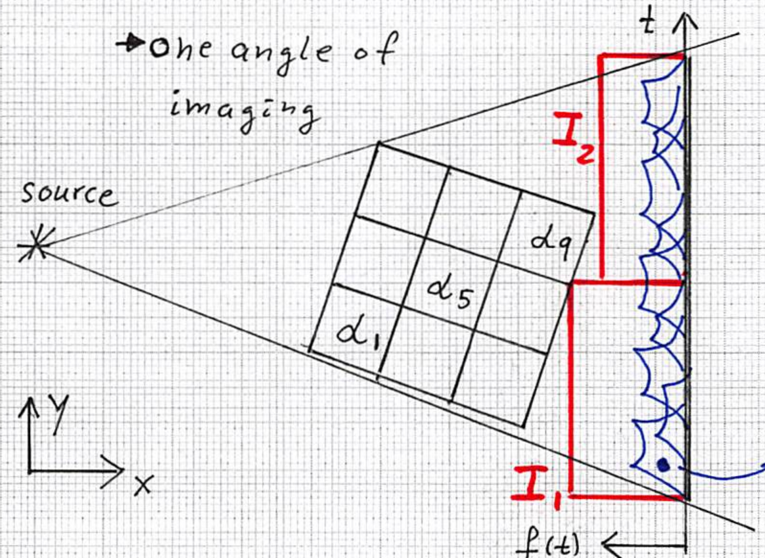
POINT LIGHT SOURCE - Cont'd.

- Practical issues: sampling and resolution

→ One angle of imaging



$N = 9$ grid resolution

$R = 2$ resolution of projection line

$A = 5$ A angles used for imaging

Very HIGH-RESOLUTION representation of $\{f_i(t)\}$

$\Rightarrow$ PRE-COMPUTED AND STORED for all A angles!

→ call the imaging angles

$\beta_1, \beta_2, \beta_3, \beta_4, \beta_5$

$\Rightarrow$ LINEAR EQUATION SYSTEM - BEST APPROXIMATION:

$$
\underbrace{\begin{array}{c} \beta_1 \left\{ \begin{array}{c} \\ \\ \\ \end{array} \right. \\ \vdots \\ \beta_5 \left\{ \begin{array}{c} \\ \\ \\ \end{array} \right. \end{array}}_{9 \cdot 5 = 45 \text{ rows}}
\begin{bmatrix}
\langle f_1, f_1 \rangle & \dots & \langle f_1, f_9 \rangle \\
\vdots & & \vdots \\
\langle f_9, f_1 \rangle & \dots & \langle f_9, f_9 \rangle \\
 & \vdots & \\
\langle f_1, f_1 \rangle & \dots & \langle f_1, f_9 \rangle \\
\vdots & & \vdots \\
\langle f_9, f_1 \rangle & \dots & \langle f_9, f_9 \rangle
\end{bmatrix}
\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_9 \end{bmatrix}
=
\begin{bmatrix}
\langle f, f_1 \rangle \\ \vdots \\ \langle f, f_9 \rangle \\ \vdots \\ \langle f, f_1 \rangle \\ \vdots \\ \langle f, f_9 \rangle
\end{bmatrix}
$$

recorded values (= piecewise const. fct.), I_1 and I_2

$$\Rightarrow \quad M\alpha = F \quad \xrightarrow[\text{squares}]{\text{Least}} \quad \underline{\underline{\alpha = (M^T M)^{-1} M^T F}}$$

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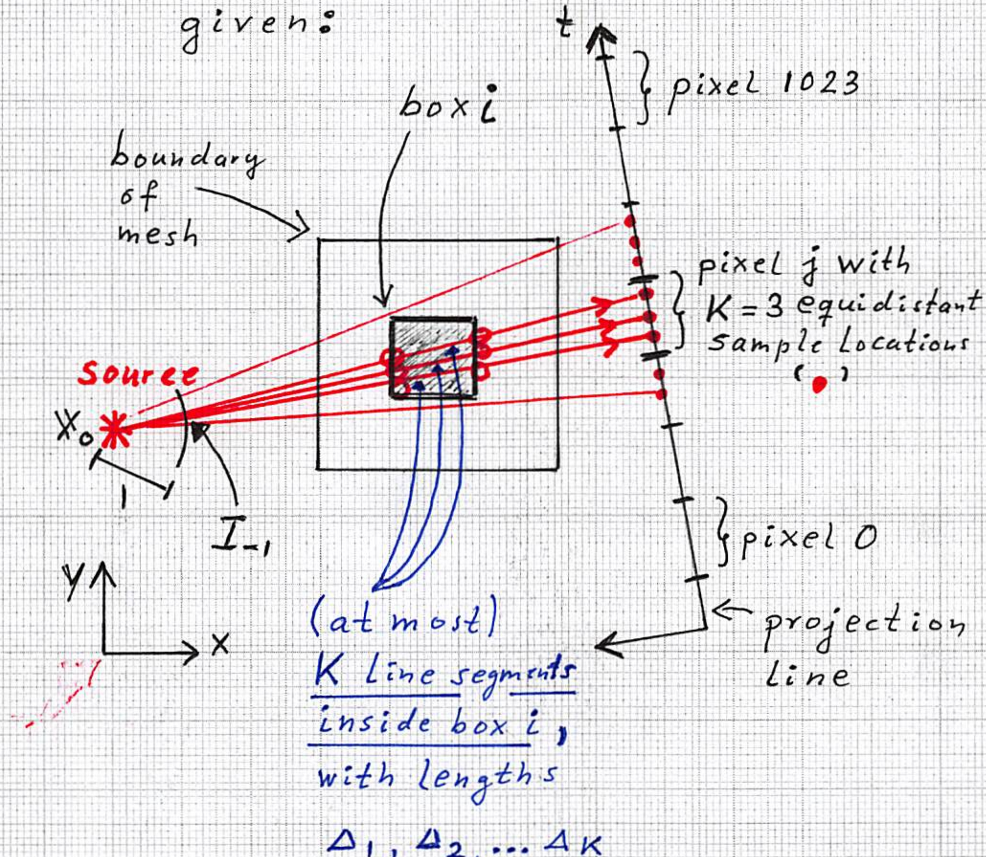
■ POINT LIGHT SOURCE – Cont'd.

- Computing – **PRE-COMPUTING** – projections of all unit box functions:
  - ➔ The exact analytical determination and representation for persective unit box functions is difficult and expensive!
  - ➔ Compute the needed unit box function projections by sampling into the grid for which the measured projection data are given:



- **Example:**
  - image resolution = 1024
  - $K=3$ = "per-pixel sampling resolution"
  - compute $K=3$ intensity values for each pixel j in which box i projects:

$$I_1 = \frac{1}{d_1} I_{-1} e^{-\Delta_1},$$

...

$$I_3 = \frac{1}{d_3} I_{-1} e^{-\Delta_3},$$

where d_1, d_2, d_3 = distances between x_0 and sample locations

$$\Rightarrow \boxed{I(i,j) = \frac{1}{K}\sum_{k=1}^{K} I_k = \frac{1}{3}\sum_{k=1}^{3} I_k}$$

= intensity of unit box function b_i projected into pixel j.

■ POINT LIGHT SOURCE – Cont'd.

➔ The linear system to be solved:

Example: • 720 imaging angles,

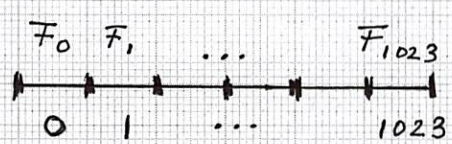
1024 pixel values recorded per angle

$\Rightarrow 720 \times 1024 = 737280$ (data)

• 2D image of resolution

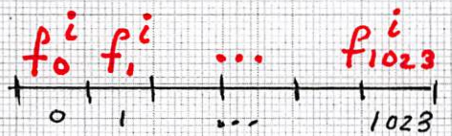
$512 \times 512 = 262144$ (boxes)

For 720 angles:



$\Rightarrow$ measured, given intensity values: $F_0, \ldots, F_{1023}$

$\Rightarrow$ vector $\mathbb{F} = \begin{pmatrix} F_0 \\ \vdots \\ F_{1023} \end{pmatrix}$



$\Rightarrow$ pre-computed and known intensity values for box i, projected into 1024 intervals

$\Rightarrow$ vector $\mathbb{f}^i = \begin{pmatrix} f_0^i \\ \vdots \\ f_{1023}^i \end{pmatrix}$

$\Downarrow$

$$\mathbb{F} = \sum_{i=1}^{512^2} \alpha_i \, \mathbb{f}^i$$

• Resulting system:

$$\begin{array}{l} \text{angle } 1 \left\{ \right. \\ \\ \\ \text{angle } 720 \left\{ \right. \end{array}
\begin{bmatrix} f_0^1 & \cdots & f_0^{512^2} \\ \vdots & & \\ f_{1023}^1 & \cdots & f_{1023}^{512^2} \\ & \vdots & \\ f_0^1 & \cdots & f_0^{512^2} \\ \vdots & & \\ f_{1023}^1 & \cdots & f_{1023}^{512^2} \end{bmatrix}
\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_{512^2} \end{bmatrix}
=
\begin{bmatrix} F_0 \\ \vdots \\ F_{1023} \\ \vdots \\ F_0 \\ \vdots \\ F_{1023} \end{bmatrix}
\begin{array}{l} \left. \right\} \text{angle } 1 \\ \\ \\ \left. \right\} \text{angle } 720 \end{array}$$

$$M \cdot \vec{\alpha} = \vec{\mathbb{F}}$$

$\Rightarrow$ solve over-determined system:

$$\boxed{M^T M \vec{\alpha} = M^T \vec{\mathbb{F}}}$$

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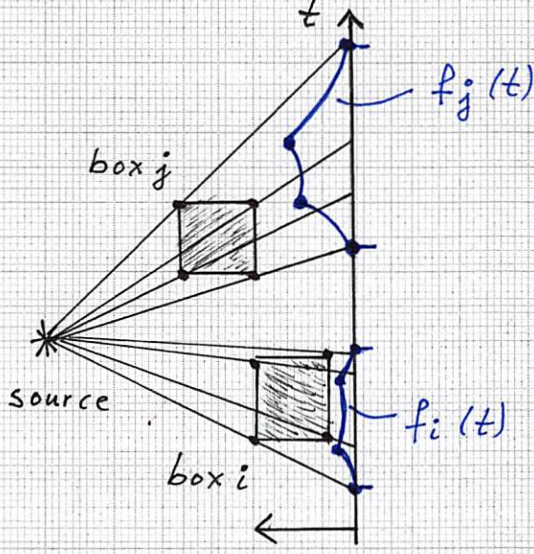
▪ POINT LIGHT SOURCE – Cont'd.

➔ Reconstruction with analytical representations of projections of unit box functions:



• assumption: projections of unit box functions known analytically ⇒ BEST APPROXIMATION of the given "1024 values" (= piecewise constant function F):

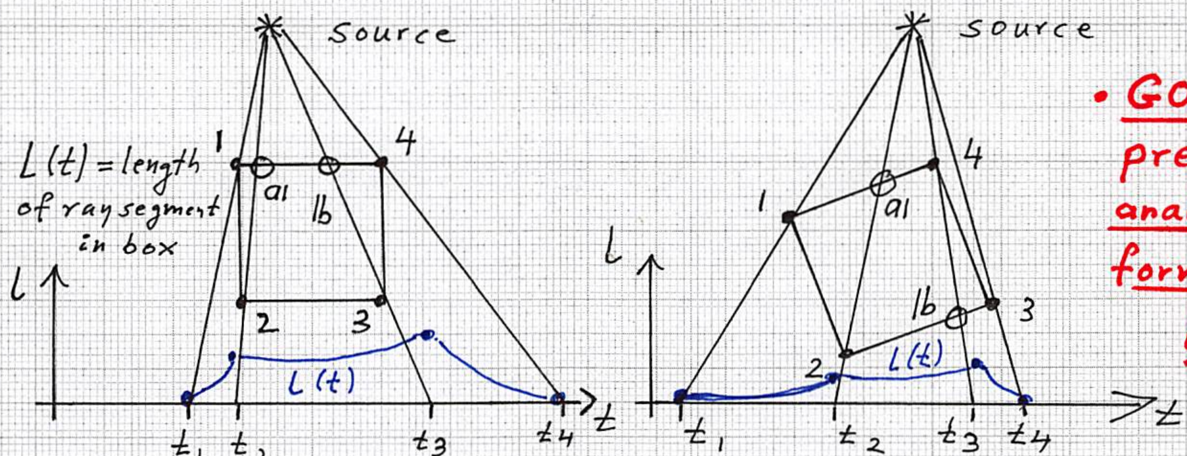
$$F(t) = \sum_{i=1}^{512^2} \alpha_i f_i(t)$$

obtained by solving

considering all data for all projection angles! ⇒

$$\begin{pmatrix} \langle F, f_1 \rangle \\ \vdots \\ \langle F, f_{512^2} \rangle \end{pmatrix} = \begin{pmatrix} \langle f_1, f_1 \rangle & \dots & \langle f_1, f_{512^2} \rangle \\ & \vdots & \\ \langle f_{512^2}, f_1 \rangle & \dots & \langle f_{512^2}, f_{512^2} \rangle \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_{512^2} \end{pmatrix}$$

➔ Qualitative nature of box projections:



• **GOAL: precise analytical form of L(t)!**

⇒ (Except in degenerate cases,) ray segment pass through two triangles and one quadrilateral: either 12a1, a1231b, 1b34 or 12a1, a121b4, 41b3.

~ BH