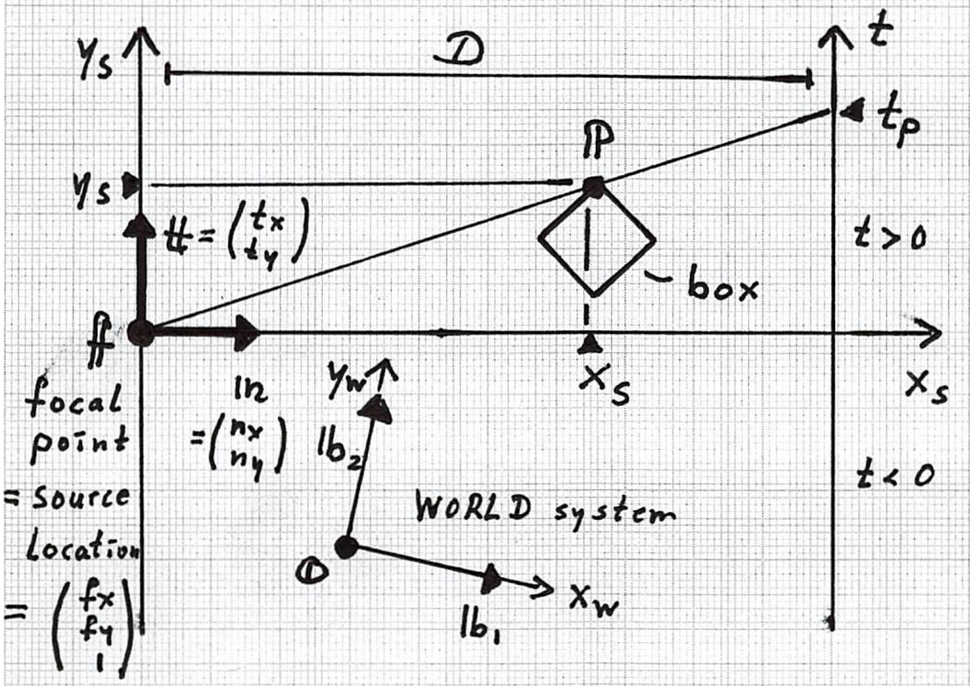
Bernd Hamann 2/19/2020

Stratovan

ANALYTICAL METHODS FOR RECONSTRUCTION – POINT LIGHT SOURCES

• Coordinate System Transformation



→ t-line = projection line

→ world coord. system: O, b_1, b_2

→ source coord. system: f, n, t

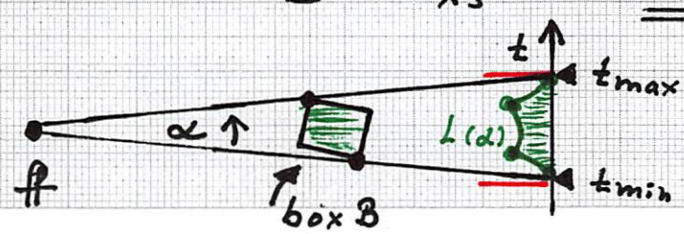
→ $P_w = \begin{pmatrix} x_w \\ y_w \\ 1 \end{pmatrix}$, $P_s = \begin{pmatrix} x_s \\ y_s \\ 1 \end{pmatrix}$

i) World-to-source transformation:

$$P_s = \begin{pmatrix} n_x & n_y & 0 \\ t_x & t_y & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -f_x \\ 0 & 1 & -f_y \\ 0 & 0 & 1 \end{pmatrix} P_w$$

ii) Projection onto t-line:

$$\frac{t_p}{D} = \frac{y_s}{x_s} \curvearrowright t_p = D\,\frac{y_s}{x_s}, \quad x_s \neq 0;\ \text{e.g.},\ D = 1$$



iii) Projection interval:

Box B "projects to" t-interval $[t_{min}, t_{max}]$ for which its projection $L(\alpha)$, $L(t)$ is defined.

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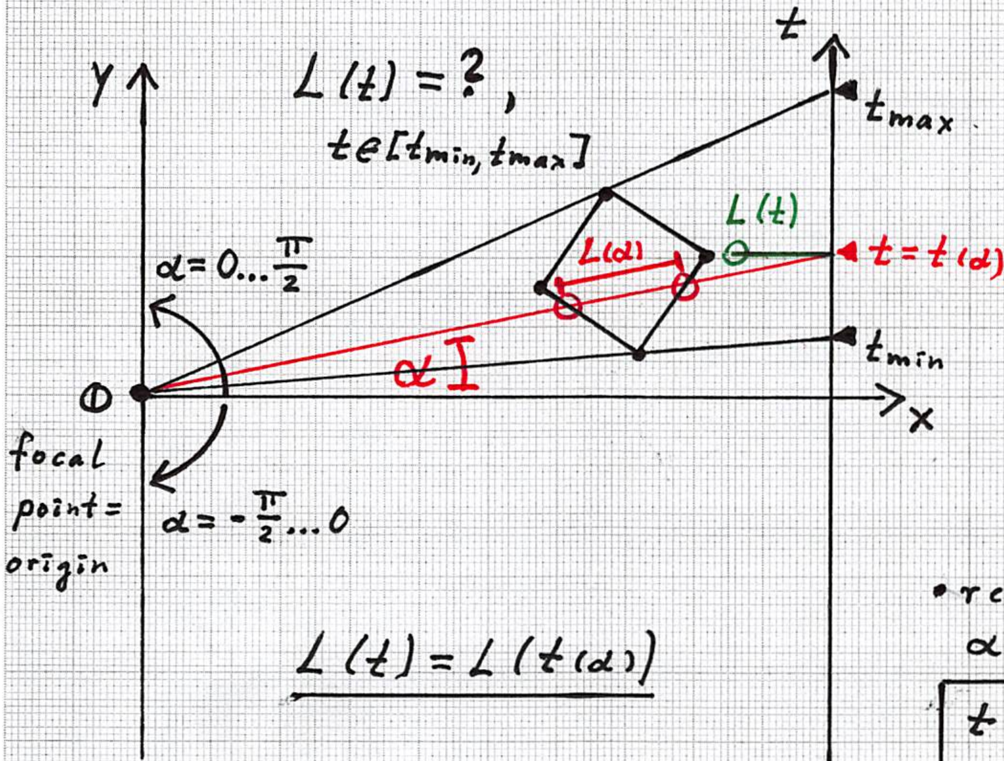
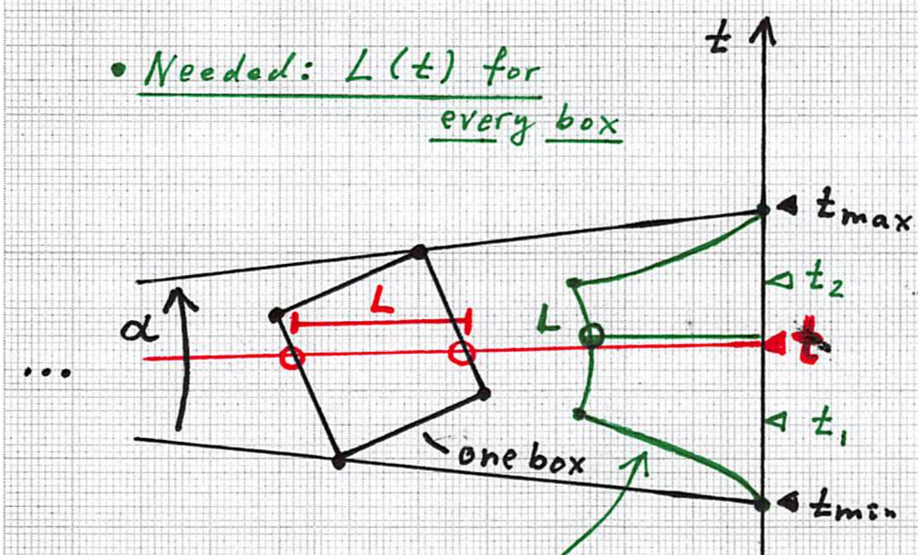
▪ ANALYTICAL METHODS – Cont'd



- all points represented relative to source coord. system
- needed: ANALYTICAL definition of $L(\alpha)$, $L(t)$
- relationship between α and t:

$$\boxed{\begin{aligned} t &= \tan \alpha \\ \alpha &= \tan^{-1} t \end{aligned}}$$

- Needed: $L(t)$ for every box



- Goal: exact analytical definition of "length function" $L(t)$ for every box, using box-specific intervals $[t_{min}, t_{max}]$

- Can $L(t)$ be defined in a piecewise, C^0-continuous way? Here: $L(t)$ has 3 pieces, for sub-intervals $[t_{min}, t_1]$, $[t_1, t_2]$, $[t_2, t_{max}]$.

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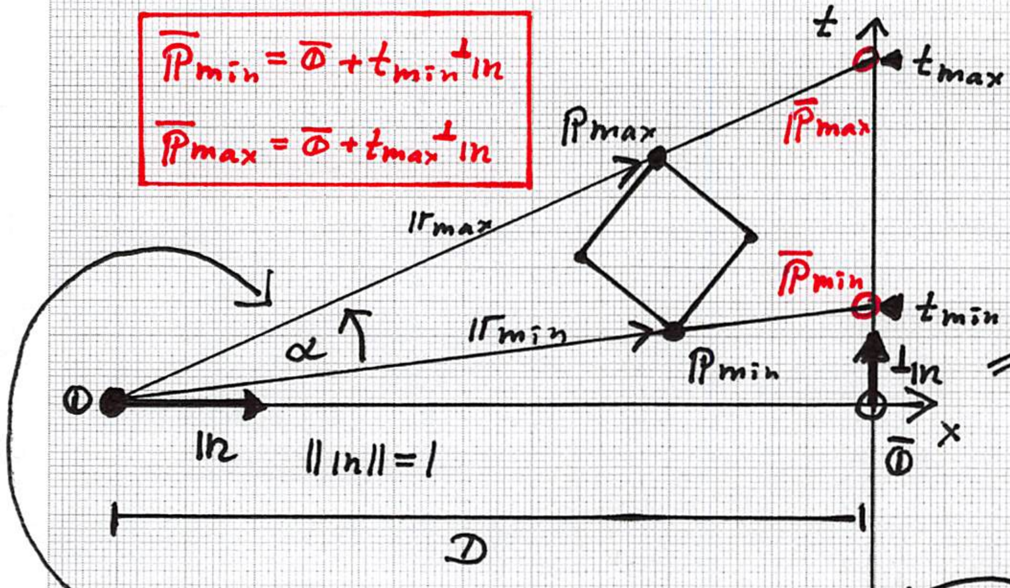
ANALYTICAL METHODS – Cont'd.

- **Mapping extremal box corners to projection line:**

$$\boxed{\begin{aligned}\overline{\mathbb{p}}_{min} &= \overline{\mathbb{O}} + t_{min}\,{}^{\perp}\mathbb{n}\\ \overline{\mathbb{p}}_{max} &= \overline{\mathbb{O}} + t_{max}\,{}^{\perp}\mathbb{n}\end{aligned}}$$



$$\underline{\mathbb{n} = \begin{pmatrix} n_x \\ n_y \end{pmatrix}}$$

$$\mathbb{n}^{\perp} = \begin{pmatrix} n_y \\ -n_x \end{pmatrix}$$

$$\underline{{}^{\perp}\mathbb{n} = \begin{pmatrix} -n_y \\ n_x \end{pmatrix}}$$

$$\underline{\overline{\mathbb{O}} = D\,\mathbb{n}}$$

IMPLICIT def. of projection line:

$$\mathbb{n} \bullet (\mathbb{x} - \overline{\mathbb{O}}) = 0 \qquad (**)$$

Parametric def. of two extremal rays $\mathbb{r}_{min}$, $\mathbb{r}_{max}$:

$$\mathbb{r}_{min} = u \cdot \mathbb{p}_{min}, \qquad (*)$$

$$\mathbb{r}_{max} = v \cdot \mathbb{p}_{max}$$

$\Rightarrow$ insert $(*)$ into $(**)$:

$\mathbb{r}_{min}$:

$$\mathbb{n} \bullet \left(u \cdot \mathbb{p}_{min} - \overline{\mathbb{O}} \right) = 0$$

$$\mathbb{n} \bullet \left(u \cdot \mathbb{p}_{min} \right) = \mathbb{n} \bullet \overline{\mathbb{O}}$$

$$u \cdot \mathbb{n} \bullet \mathbb{p}_{min} = \mathbb{n} \bullet \overline{\mathbb{O}}$$

$$\tilde{u} = \frac{\mathbb{n} \bullet \overline{\mathbb{O}}}{\mathbb{n} \bullet \mathbb{p}_{min}} \qquad \Rightarrow \overline{\mathbb{p}}_{min} = \tilde{u}\,\mathbb{p}_{min}$$

$\mathbb{r}_{max}$:

$$\tilde{v} = \frac{\mathbb{n} \bullet \overline{\mathbb{O}}}{\mathbb{n} \bullet \mathbb{p}_{max}} \qquad \Rightarrow \overline{\mathbb{p}}_{max} = \tilde{v}\,\mathbb{p}_{max}$$

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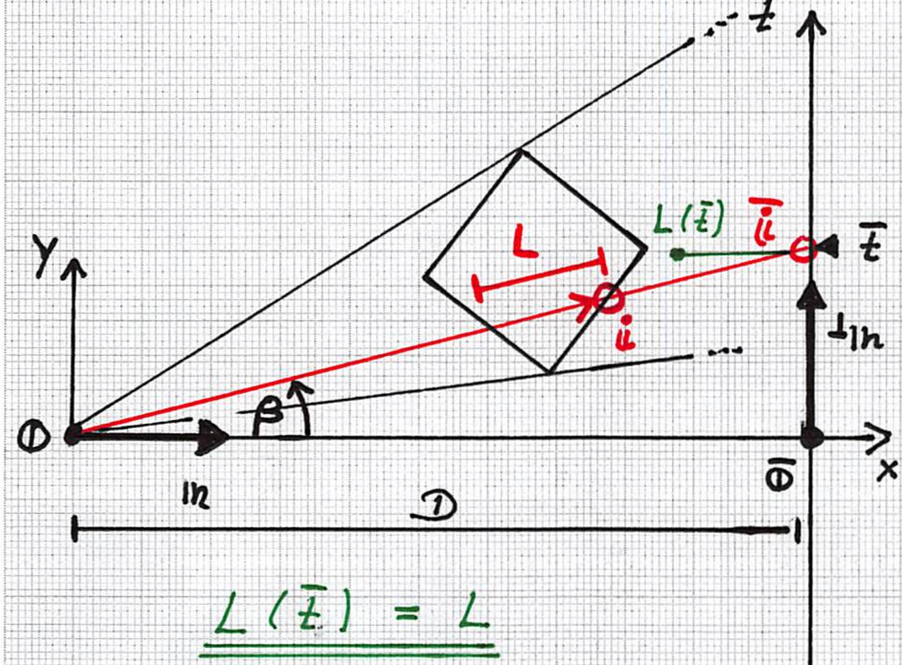
ANALYTICAL METHODS – Cont'd.

- Mapping box edge intersection points to projection line:



$$\underline{\underline{L(\bar t) = L}}$$

$$\underline{\bar{\mathbb{i}} = \bar{\mathbb{o}} + \bar t \perp\mathbb{m}},$$

where:

$\bar{\mathbb{i}} = \bar u \cdot \mathbb{i}$ and

$$\bar u = \frac{\mathbb{m} \bullet \bar{\mathbb{o}}}{\mathbb{m} \bullet \mathbb{i}}$$

$\Rightarrow \bar t$ defined by

$$\underline{\underline{\bar t \perp\mathbb{m} = \bar{\mathbb{i}} - \bar{\mathbb{o}}}}$$

(use x- or y-coord. to compute $\bar t$ value.)

- Using tan rule:

$$\tan\beta = \bar t / D \Rightarrow \underline{\underline{\bar t = D \tan\beta}},$$

where

$$\underline{\cos\beta} = \frac{\mathbb{m} \bullet \mathbb{i}}{\|\mathbb{m}\|\,\|\mathbb{i}\|} = \underline{\frac{\mathbb{m} \bullet \mathbb{i}}{\|\mathbb{i}\|}}$$

$$\Rightarrow \beta = \ldots$$

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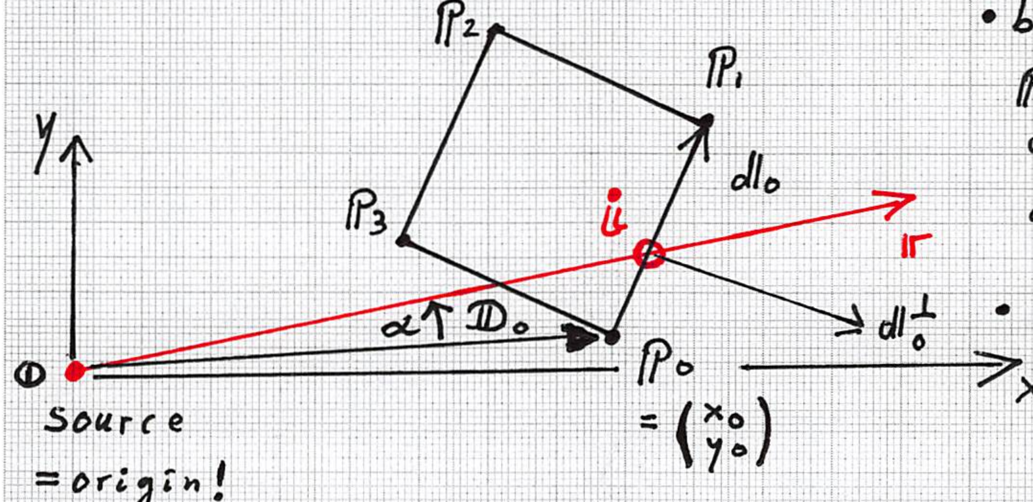
■ ANALYTICAL METHODS - Cont'd.

- Intersection of ray and box edge:



- box with corners $\mathbb{P}_0, \mathbb{P}_1, \mathbb{P}_2, \mathbb{P}_3$ and edge vectors $\mathbf{d}_0, \mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3$

- $\underline{\mathbb{D}_i = \mathbb{P}_i - \mathbf{O}}$
$= \underline{\mathbb{P}_i}$,
$\mathbb{D}_i = \begin{pmatrix} D_x^i \\ D_y^i \end{pmatrix}$

i) Parametric ray definition:

$$\underline{\underline{\mathbf{r} = \mathbf{r}(u,\alpha) = u \cdot R(\alpha) \cdot \overset{=\mathbb{P}_0}{\mathbb{D}_0}}} = u \cdot \begin{pmatrix} c\alpha & -s\alpha \\ s\alpha & c\alpha \end{pmatrix} \overset{\mathbb{P}_0 =}{\begin{pmatrix} D_x^0 \\ D_y^0 \end{pmatrix}}$$

$$= \begin{pmatrix} x(u,\alpha) \\ y(u,\alpha) \end{pmatrix} \quad (*)$$

ii) Implicit edge definition:

$$\mathbf{d}_0 = \mathbb{P}_1 - \mathbb{P}_0 = \begin{pmatrix} d_x^0 \\ d_y^0 \end{pmatrix} , \quad \mathbf{d}_0^{\perp} = \begin{pmatrix} d_y^0 \\ -d_x^0 \end{pmatrix}$$

$$\Rightarrow \mathbf{d}_0 \bullet \mathbf{d}_0^{\perp} = 0$$

$\Rightarrow$ Line containing edge $\overline{\mathbb{P}_0 \mathbb{P}_1}$ defined as:

$$\underline{\underline{\mathbf{d}_0^{\perp} \bullet (\mathbf{x} - \mathbb{P}_0) = 0}} \rightarrow \mathbf{d}_0^{\perp} \bullet \mathbf{x} = \mathbf{d}_0^{\perp} \bullet \mathbb{P}_0$$

$$\overset{\text{use}}{\underset{(*)}{\Longrightarrow}} \mathbf{d}_0^{\perp} \bullet \left(u \cdot R(\alpha) \cdot \overset{=\mathbb{P}_0}{\mathbb{D}_0} \right) = \mathbf{d}_0^{\perp} \bullet \mathbb{P}_0$$

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■ ANALYTICAL METHODS – Cont'd.

$$\Rightarrow \ldots \quad u \cdot \mathbf{d}_0^{\perp} \bullet R(\alpha) \cdot \overset{=\mathbf{p}_0}{\mathbf{D}_0} = \mathbf{d}_0^{\perp} \bullet \mathbf{p}_0$$

$$\curvearrowright \quad \bar{u} = \frac{\mathbf{d}_0^{\perp} \bullet \mathbf{p}_0}{\mathbf{d}_0^{\perp} \bullet R(\alpha) \cdot \mathbf{D}_0} \overset{\mathbf{D}_0 = \mathbf{p}_0}{=} \frac{\mathbf{d}_0^{\perp} \bullet \mathbf{p}_0}{\mathbf{d}_0^{\perp} \bullet R(\alpha) \cdot \mathbf{p}_0}$$

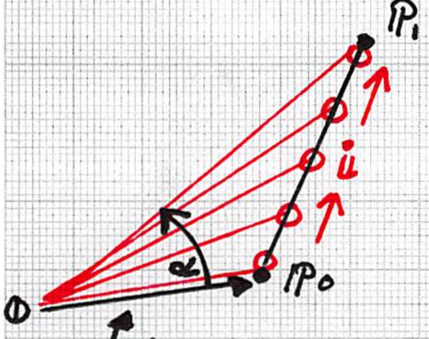
$\Rightarrow$ intersection point between ray and line containing edge $\overline{\mathbf{p}_0 \mathbf{p}_1}$: $\mathbf{i} = \bar{u} \cdot R(\alpha) \cdot \mathbf{p}_0$

(• Restrict α to range $[0, \measuredangle(\vec{\mathbf{p}}_0, \vec{\mathbf{p}}_1)]$!

$\Rightarrow$ $\mathbf{i}$ lies inside edge $\overline{\mathbf{p}_0 \mathbf{p}_1}$.)

{ • **NOTE:** $R(\alpha) = \begin{pmatrix} c\alpha & -s\alpha \\ s\alpha & c\alpha \end{pmatrix}$

$$\Rightarrow \bar{u} = \frac{\mathbf{d}_0^{\perp} \bullet \mathbf{p}_0}{\mathbf{d}_0^{\perp} \bullet \begin{pmatrix} x_0 c\alpha - y_0 s\alpha \\ x_0 s\alpha + y_0 c\alpha \end{pmatrix}}$$



$\vec{\mathbf{p}}_0 = \mathbf{p}_0 - \mathbf{0} = \mathbf{p}_0$

$$= \frac{\mathbf{d}_0^{\perp} \bullet \mathbf{p}_0}{d_y^0 (x_0 c\alpha - y_0 s\alpha) + (-d_x^0)(x_0 s\alpha + y_0 c\alpha)}$$

$$= \frac{\mathbf{d}_0^{\perp} \bullet \mathbf{p}_0}{c\alpha (d_y^0 x_0 - d_x^0 y_0) + s\alpha (-d_y^0 y_0 - d_x^0 x_0)}$$

$$= \frac{\mathbf{d}_0^{\perp} \bullet \mathbf{p}_0}{c\alpha \cdot \mathbf{d}_0^{\perp} \bullet \mathbf{p}_0 - s\alpha \cdot \mathbf{d}_0 \bullet \mathbf{p}_0}$$

$$\Rightarrow \mathbf{i} = \bar{u} \cdot R(\alpha)\, \mathbf{p}_0 = \bar{u} \cdot \begin{pmatrix} x_0 c\alpha - y_0 s\alpha \\ x_0 s\alpha + y_0 c\alpha \end{pmatrix} \quad \}$$

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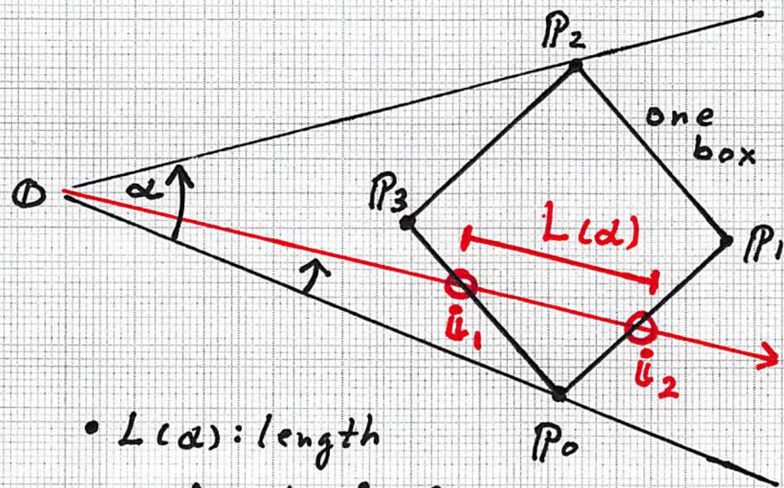
ANALYTICAL METHODS – Cont'd.

- Computation of pairs of intersection points between rays and box edges:



- compute the two points $\mathring{u}_1$ and $\mathring{u}_2$, using ray-edge intersection method

- $L(\alpha)$: length of vector $\mathring{u}_2 - \mathring{u}_1$

- $\mathring{u}_1 = \bar{u}_1 \cdot \sim$

  $\mathring{u}_2 = \bar{u}_2 \cdot \sim$

$$\Rightarrow \underline{\underline{L(\alpha)}} = \| \mathring{u}_2 - \mathring{u}_1 \| = \| \bar{u}_2 \cdot \sim - \bar{u}_1 \cdot \sim \|$$

$$= \underline{\underline{| \bar{u}_2 - \bar{u}_1 | \cdot \| R(\alpha) \cdot \mathbb{p}_0 \|}}$$

(here: use $R(\alpha)\, \mathbb{p}_0$ since $\mathring{u}_1$ and $\mathring{u}_2$ lie on edges $\overline{\mathbb{p}_0 \mathbb{p}_3}$ and $\overline{\mathbb{p}_0 \mathbb{p}_1}$, respectively.)

- Cases to be handled: I, II, III, IV $\Rightarrow C^0$-continuous $L(\alpha)$

- The box regions T (triangle), Q (quadrilateral) define the 3 (or 2) pieces of $L(\alpha)$.