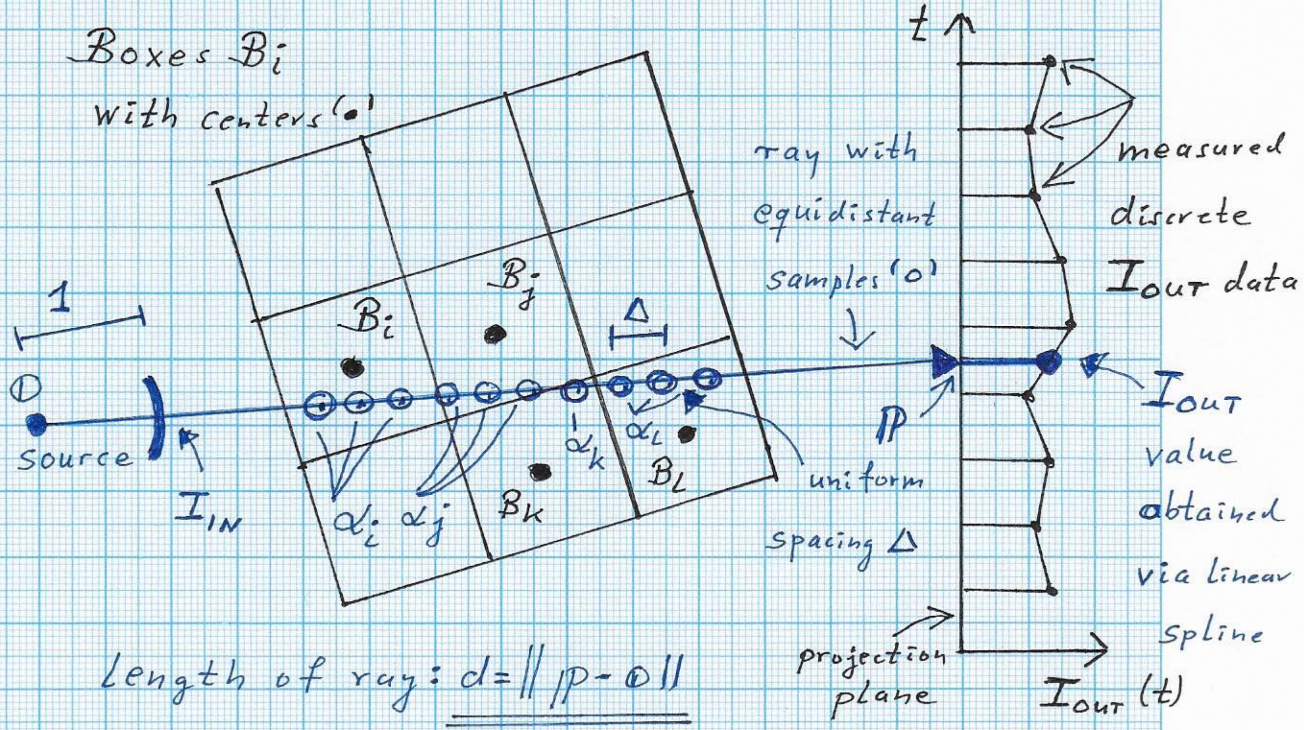
Bernd Hamann — 1 — 4/6/2020

Stratovan

■ AN APPROXIMATE RECONSTUCTION METHOD

– FOR PARALLEL & PERSPECTIVE PROJECTION



length of ray: $d = \| \mathbb{P} - \mathbb{O} \|$

- Sample points 'o' in same box have same α-value.
- Box B_i will be assigned the α-value of all sample points 'o' lying in box b_i, considering ALL rays.
- Above example: (eq. for 3D case) (perspective case considering d^2)

$$\Delta(3\alpha_i + 3\alpha_j + 1\alpha_k + 3\alpha_L) = -\ln\left(d^2 \frac{I_{OUT}}{I_{IN}}\right)$$

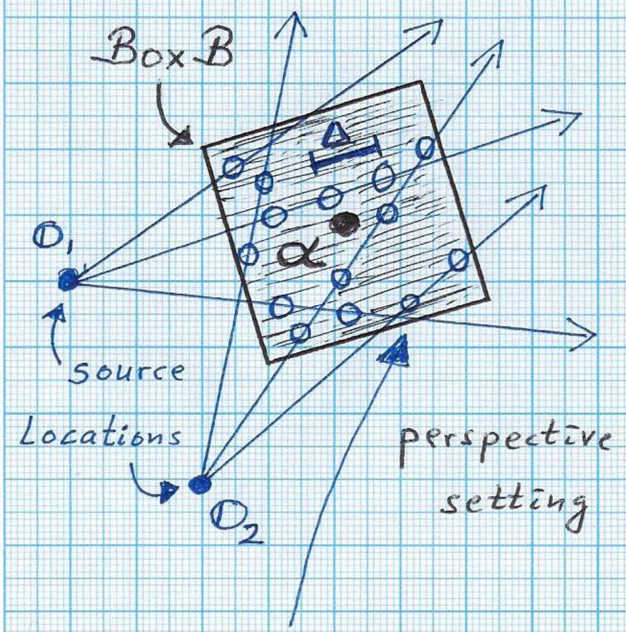
- no. unknown α-values = no. of boxes
- Start with small no. of boxes and then increase no. of boxes ⇒ MULTI-RESOLUTION (possibly use adaptive refinement: quadtree)

Bernd Hamann 4/7/2020

Stratovan

■ APPROXIMATE RECONSTRUCTION... Cont'd.

• "Quasi Monte Carlo" (QMC) method:



→ Compute ONE α-value for each box B.

→ Ray passing through B define set of sample points {o}.

→ Grid consisting of boxes {B} is fixed

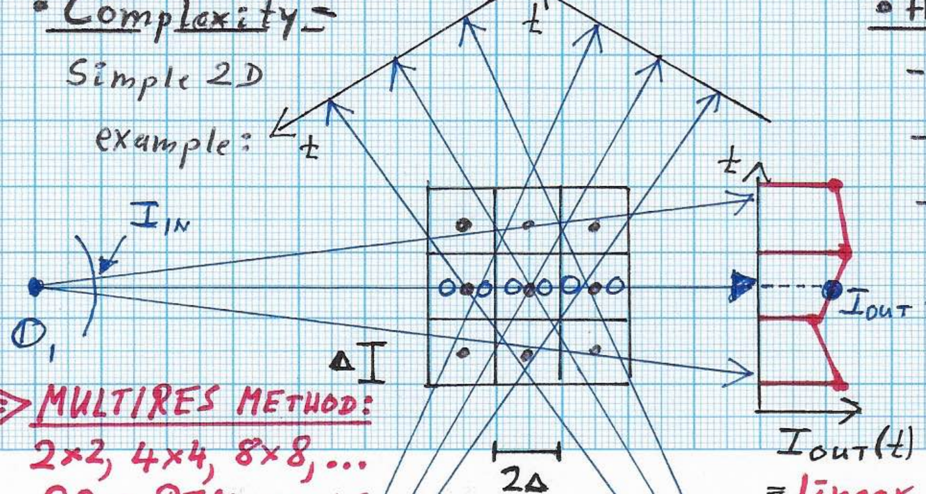
→ Different projections are obtained by different source locations {O} and projection planes.

⇒ SET UP LINEAR SYSTEM FOR UNKOWN α-VALUES SUCH THAT ALL LOCATIONS 'o' HAVE THE SAME α-VALUE!

⇒ QMC method

• Complexity:

Simple 2D example:



= Linear spline

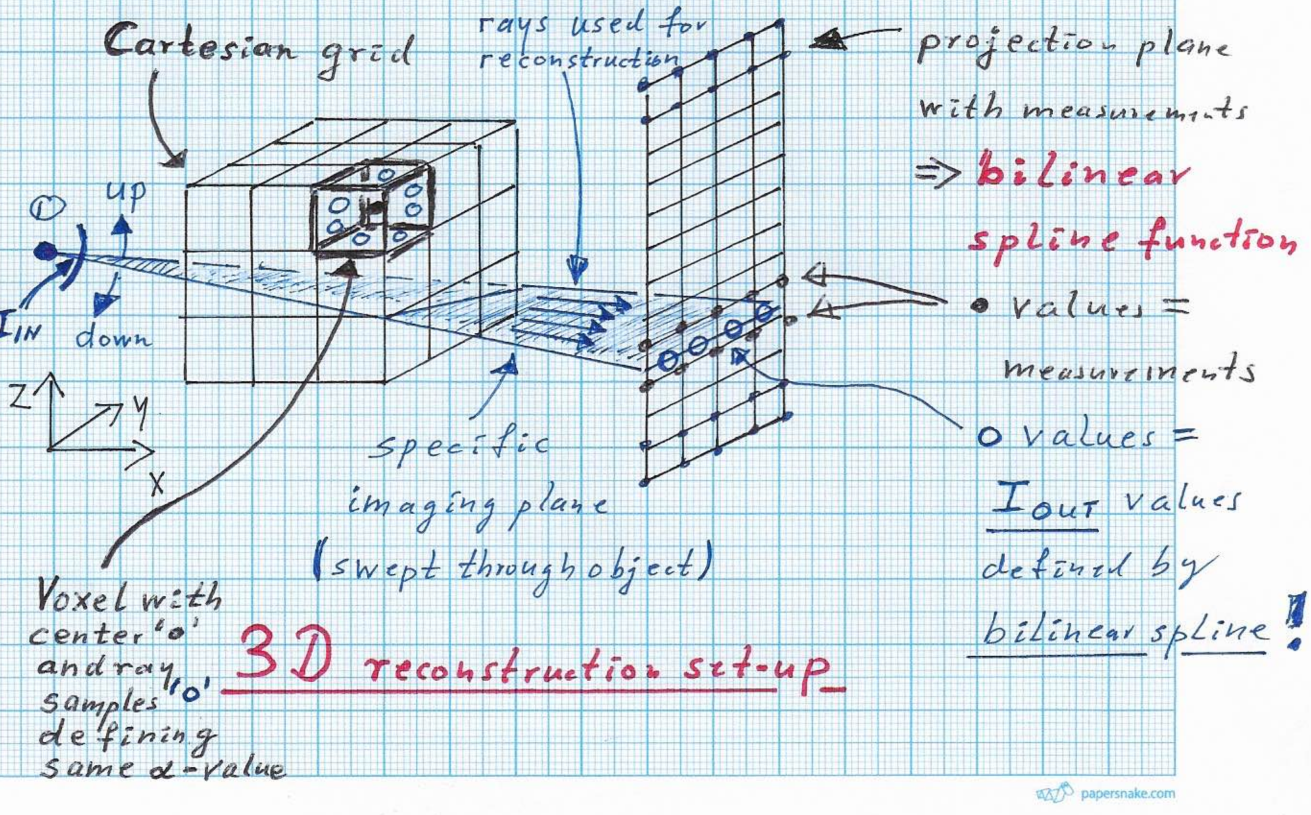
⇒ MULTIRES METHOD:
2×2, 4×4, 8×8, ... GRID RESOLUTIONS

• Here:

- grid res.: 3×3
- no. projections: 3
- no. measurements per projection: 3
- sample spacing: $\Delta = \frac{1}{2} e$
- no. of rays for defining linear sys. (per projection): 3
- unknown α-values: 9

▪ APPROXIMATE RECONSTRUCTION... Cont'd. – QMC

- Complexity – 3D case:
  - grid resolution : 512^3 (= no. unknown α-values)
  - no. of projection angles (per imaging plane through 3D object) : 720
  - no. of imaging planes swept through 3D object : P
  - no. of recorded sensor measurements per projection angle and imaging plane : 1024
  - no. of rays used for defining linear system per projection angle and imaging plane : 1024
  - no. of sample locations per ray : 1024 or 2048

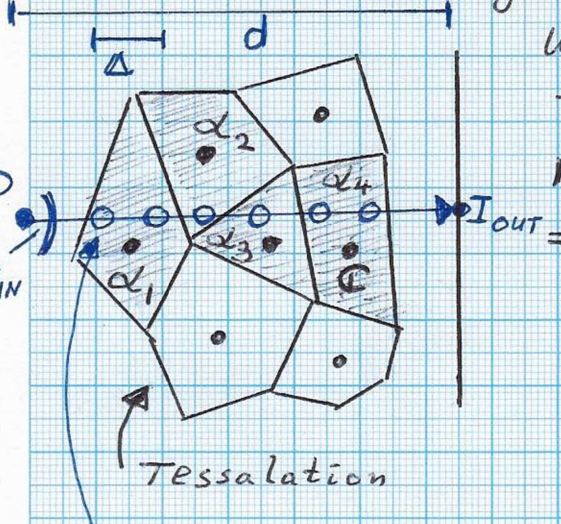


3D reconstruction set-up

Stratovan

APPROXIMATE RECONSTRUCTION... Cont'd. – QMC

→ General setting: CENTERS OF VOLUME ELEMENTS USED FOR RECONSTRUCTION ARE TILES OF A VORONOI DIAGRAM, I.E., THEY ARE TILE CENTERS.



⇒ Compute one α-value per tile. Tile centers C are assigned the α-value computed for all sample points 'o' inside the tile.

Equation for this ray:

$$\Delta\left(2\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4\right) = -\ln\left(d^2 \frac{I_{OUT}}{I_{IN}}\right)$$

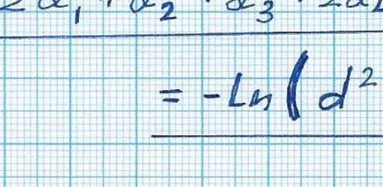
⇒ **Voronoi diagram:**
- all tiles convex
- tile containing sample 'o' simply defined: tile center '•' = nearest neighbor of 'o'

!

• No need to store VORONOI tesselation explicitly: tile containing 'o' implicitly defined!

⇒ Possibility to ADAPTIVELY PLACE TILE CENTERS – high density of tiles only in areas of high variation of α-values

Ex:



⟺ Compute α-values $\alpha_1, \dots, \alpha_N$ for these tile centers; then superimpose a high-res. Cartesian grid and define α-values for its voxels.

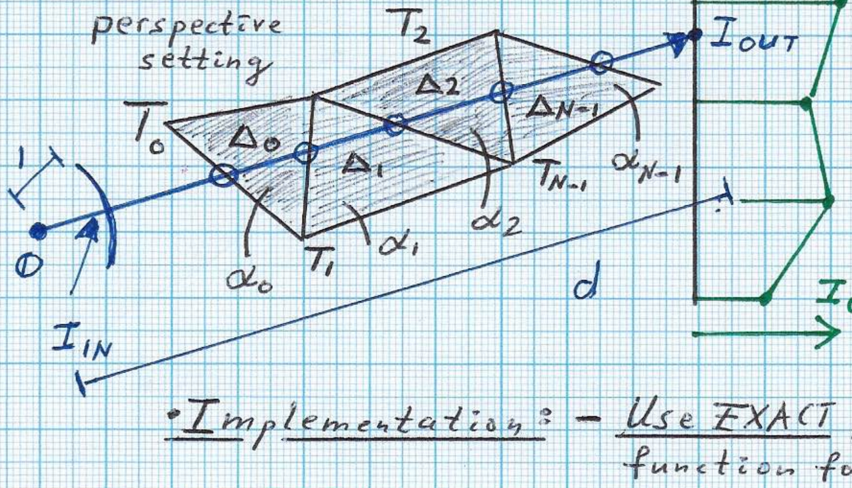
• **Note:** Is a meshless method using radial basis functions viable?

■ RECONSTRUCTION USING SIMPLICIAL GRIDS

– TRIANGULAR & TETRAHEDRAL GRIDS

• Why? – Simplest grid type
– Supporting adaptive, data-dependent placement and number of triangles/tets
– "Constant simplex basis functions" produce "simple" analytically defined basis functions on the projection line/plane.
– Number of simplices "relatively low" as density of simplices must be high only in regions of high variation.
– Computational efficiency due to "relatively low" number of simplices

• 2D case:



• Law:

$$\sum_{i=0}^{N-1} \Delta_i \alpha_i = -\ln\left(d \frac{I_{OUT}}{I_{IN}}\right)$$

d^2 in 3D

• Implementation: – Use EXACT projection of constant function for T_i on the projection line.
– Compute best approx. of recorded I_{OUT} function via least squares approx. using the exact projected function of T_i.
– Progressively and adaptively refine grid until error threshold is met.

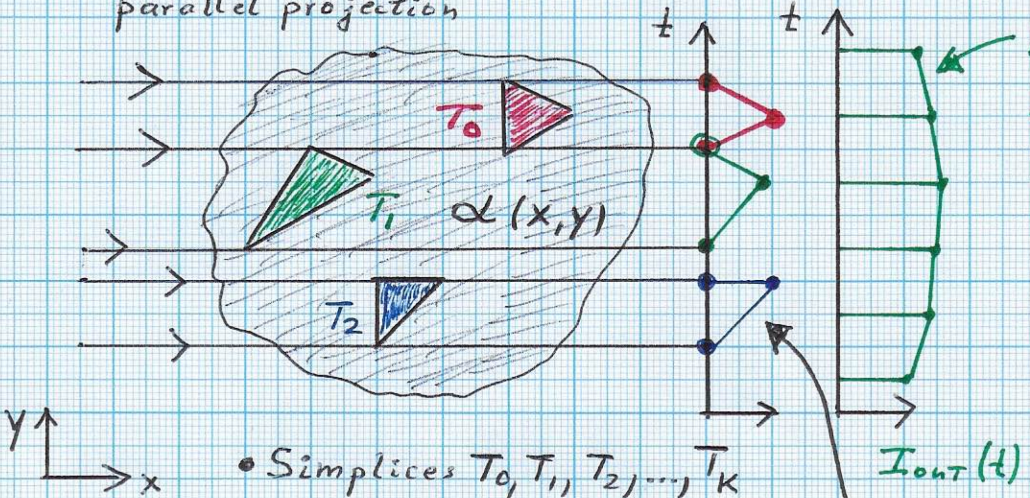
• FINAL STEP:
USE SIMPLICIAL GRID TO RE-SAMPLE ONTO A HIGH-RES CARTESIAN GRID.

Stratovan

▪ RECONSTRUCTION USING SIMPLICIAL GRIDS ... Cont'd.

• 2D case:

parallel projection



Measured I_{OUT} function: Linear spline defined by recorded intensities I_{OUT}

• Simplices $T_0, T_1, T_2, \ldots, T_K$ used to represent the region containing object

Exact, analytically defined projections of constant simplex ⊛ basis functions $I^i_{OUT}(t)$

• **Approach:** Compute best approx. of $I_{OUT}(t)$ via linear combination of $I^i_{OUT}(t)$,

$$\underline{\underline{I_{OUT} = \sum_{i=0}^{K} c_i \, I^i_{OUT}}}$$

solve normal equations:

$$\left(\langle I^i_{OUT}, I^j_{OUT} \rangle \right) \begin{pmatrix} c_0 \\ \vdots \\ c_K \end{pmatrix} = \begin{pmatrix} \langle I_{OUT}, I^1_{OUT} \rangle \\ \vdots \\ \langle I_{OUT}, I^K_{OUT} \rangle \end{pmatrix} ⊛$$

⊛ must also apply the "$-\ln$" function for $I^i_{OUT}(t)$ definition.

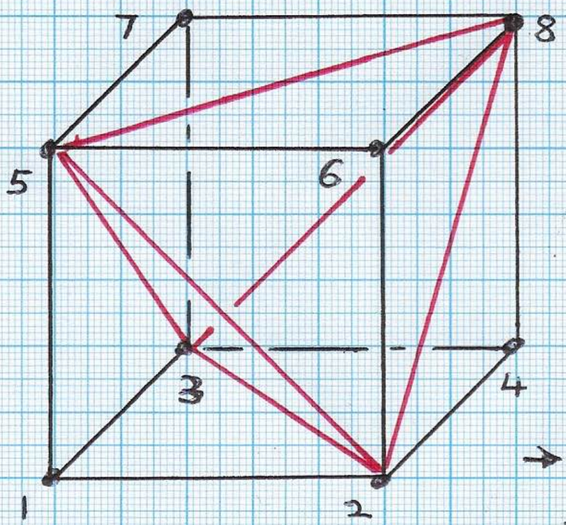
• **Note:** – Basis function $T_i(x,y)$ are constant (1) over simplices T_i.

– α-values are computed for each T_i.

– Result is $\underline{\underline{\alpha(x,y) = \sum_{i=0}^{K} \alpha_i T_i(x,y)}}$

■ RECONSTRUCTION USING SIMPLICIAL GRIDS – Cont'd.

- Why? "Simplicity of geometry"
  - → Consider 2D/3D parallel & perspective projections
- 3D Case: For example, consider subdividing each CUBE of a Cartesian mesh into FIVE TETRAHEDRA



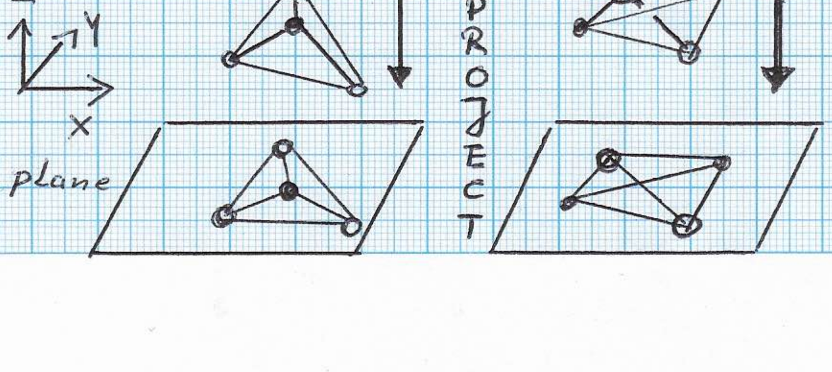
→ vertices of 5 tetrahedra:

(1, 2, 3, 5)
(2, 4, 3, 8)
(2, 5, 6, 8)
(3, 5, 7, 8)
(2, 3, 5, 8)

→ compute one constant intensity (or α) value per tetrahedron and average 5 tet values to define cube value.

⇒ BOX reconstruction simplified to tet reconstruction!

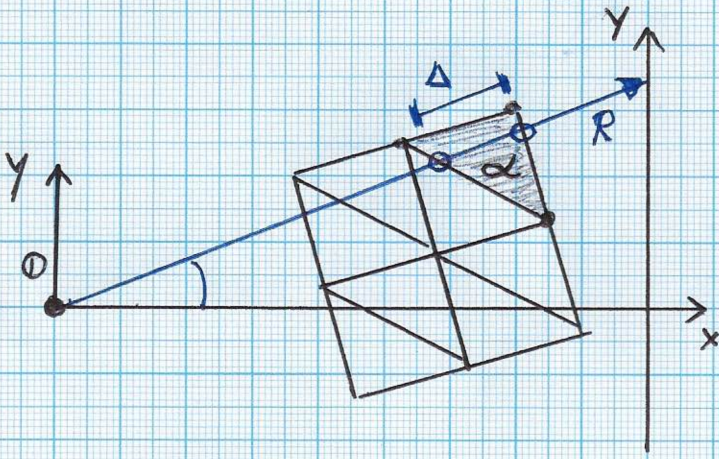
- General case: Use arbitrary tetrahedral mesh!!
- TET projections:



→ convex tet in 3D space projects to triangle or convex quadrilateral in 2D plane.

■ RECONSTRUCTION USING SIMPLICIAL GRIDS... Cont'd.

• Ex: 2D Cartesian grid handled as triangular grid; perspective projection:

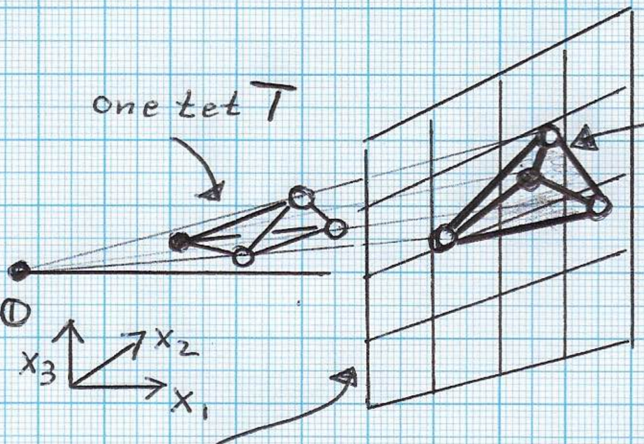


→ must perspectively project triangle (tetrahedron) onto projection line (plane);

→ must/can use exact, analytical definition of segment length Δ of ray R inside a simplex.

• 3D case:

one tet T



triangular region resulting from tet projection = DOMAIN of projected TET basis function in projection plane

projection plane with, e.g., 1024^2 recorded measurements (defining BILINEAR SPLINE).

→ must compute exact, analytical definition of ray segment length Δ.

→ TET-based piecewise constant reconstruction = SIMPLEST approx. for 3D case - supporting adaptive tet. placement & resolution/density.

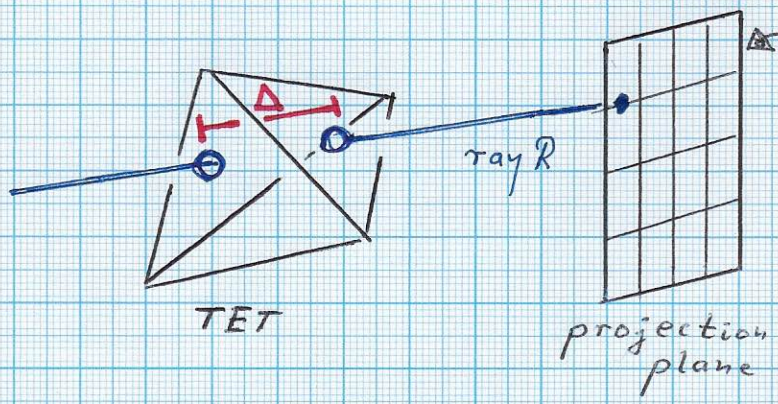
→ SPARSE LINEAR SYSTEM possible!

Bernd Hamann
Stratovan

4/14/2020

▪ RECONSTRUCTION USING SIMPLICIAL GRIDS ... Cont'd.

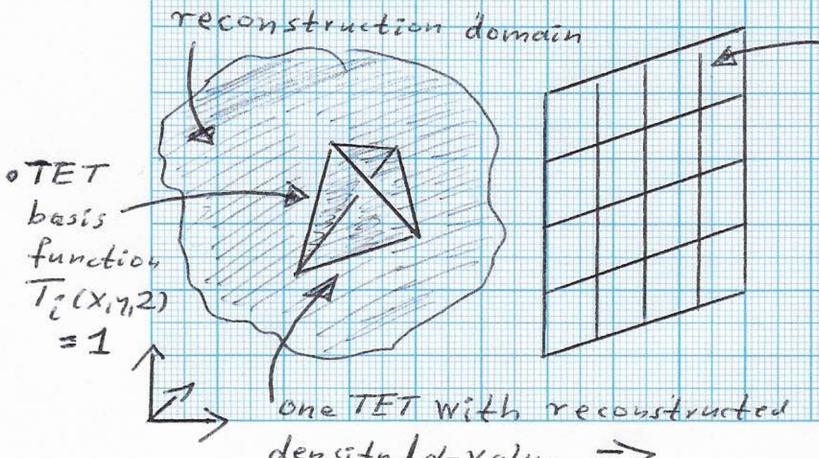
→ Construct "geometrical footprint" of every TET:



plane with measurement data ⇒ BILINEAR SPLINE definition of measurement function!

- must determine the "TET footprint projection function" in projection plane
  - requiring also exact analytical definition of ray segment length Δ inside a UNIT TET BASIS ELEMENT in 3D space

→ GOAL: BEST APPROXIMATION of recorded bilinear spline of measurements, using least squares approximation with the projected TET footprint functions in projection plane



projection plane ⇒ recorded bilinear spline!

→ use inverse distance law (for perspective projection) and absorption for reconstruction:

$$\sum_i \Delta_i \alpha_i = -\ln\left(d^2 \frac{I_{OUT}}{I_{IN}}\right)$$

one TET with reconstructed density / α-value ⇒

$$\alpha(x,y,z) = \sum_i \alpha_i T_i(x,y,z)$$

BH