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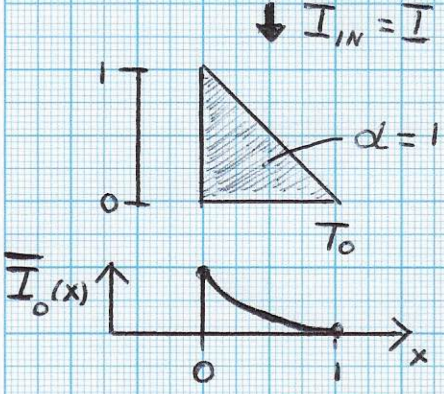
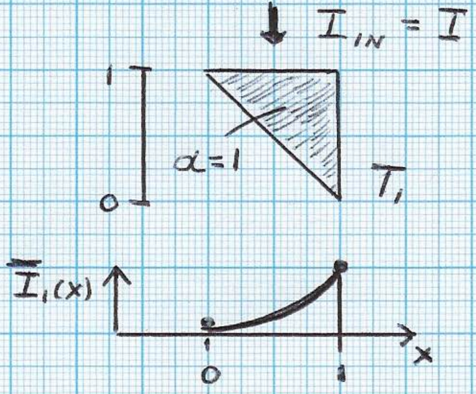
4/20/2020

■ RECONSTRUCTION USING SIMPLICIAL GRIDS ...

• Example: 2D reconstruction (parallel projection)





• 2 basis elements: T_0, T_1

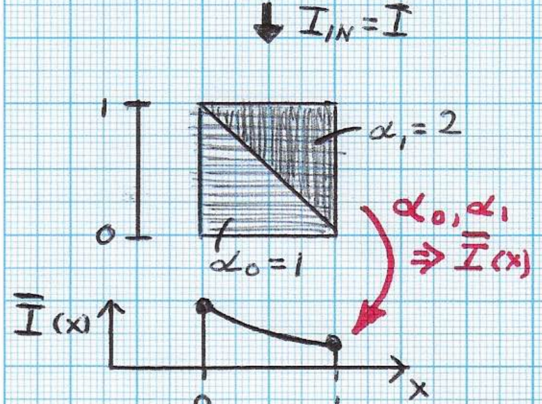
• T_0, T_1 have unit value $\alpha = 1$.

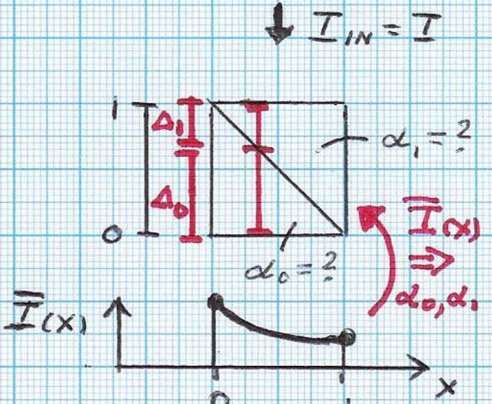
$$\overline{I}_0(x) = I e^{-(1-x)}$$

$$= \begin{cases} I/e, & x=0 \\ I, & x=1 \end{cases}$$

$$\overline{I}_1(x) = I e^{-x}$$

$$= \begin{cases} I, & x=0 \\ I/e, & x=1 \end{cases}$$





• Here: α_0, α_1 unknown; $\overline{I}(x)$ recorded

• "Exact" recording: $\overline{I}(x) = I e^{-(1+x)}$

$$\overline{I}(x) = I e^{-(1\cdot(1-x)+2x)}$$

$$= I e^{-(1+x)}$$

$$= \begin{cases} I/e, & x=0 \\ I/e^2, & x=1 \end{cases}$$

• Here: α_0, α_1 given

$$\Rightarrow \alpha_0 \Delta_0(x) + \alpha_1 \Delta_1(x) = -\ln\left(\overline{I}(x)/I\right)$$

$$= -\ln\left(e^{-(1+x)}\right) = 1+x$$

$$\Leftrightarrow \alpha_0 (1-x) + \alpha_1 x = 1+x$$

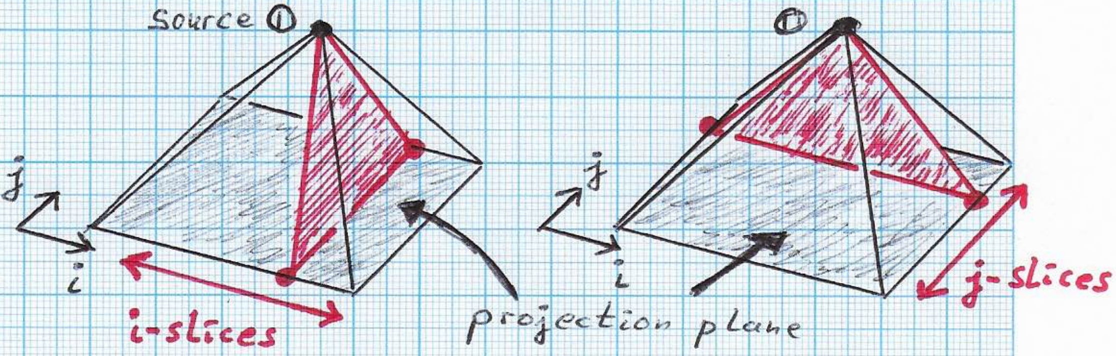
$\Rightarrow$ **BEST APPROX. OF $(1+x)$:**

$$\underline{1\cdot(1-x) + 2\cdot x = 1+x}$$

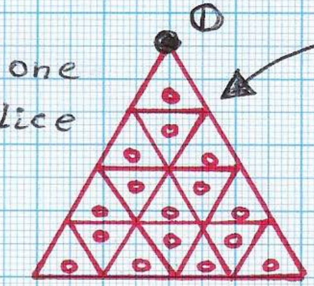
$$\Rightarrow \underline{\alpha_0 = 1, \ \alpha_1 = 2} \Rightarrow \text{Exact reconstruction}$$

▪ RECONSTRUCTION USING A SLICE-BASED APPROACH

• **IDEA:** Given projection data in the projection plane – defining a bilinear spline – compute multiple slice reconstructions. "Combine" the slice reconstructions to define the needed 3D reconstruction.

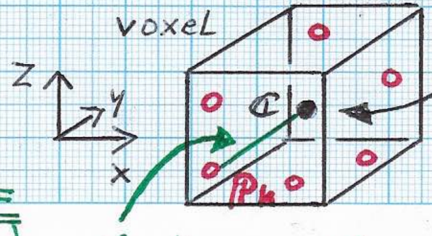


⇒ For example, use a regular triangular grid in each i-slice and j-slice to compute high-quality 2D slice reconstructions:



• Use 2D reconstruction approach for triangular grids (perspective projection) to compute slice reconstruction.

⇒ α-values computed per triangle (o).



• Compute α-value for each voxel (with center $\mathbb{C}$) via local convex combination of α-values of triangles:

$\alpha_k = \alpha(\mathbb{P}_k)$, $d_k^2 = \| \mathbb{C} - \mathbb{P}_k \|^2$

$$\alpha(\mathbb{C}) = \sum_k 1/d_k^2 \cdot \alpha_k \Big/ \sum_k 1/d_k^2$$

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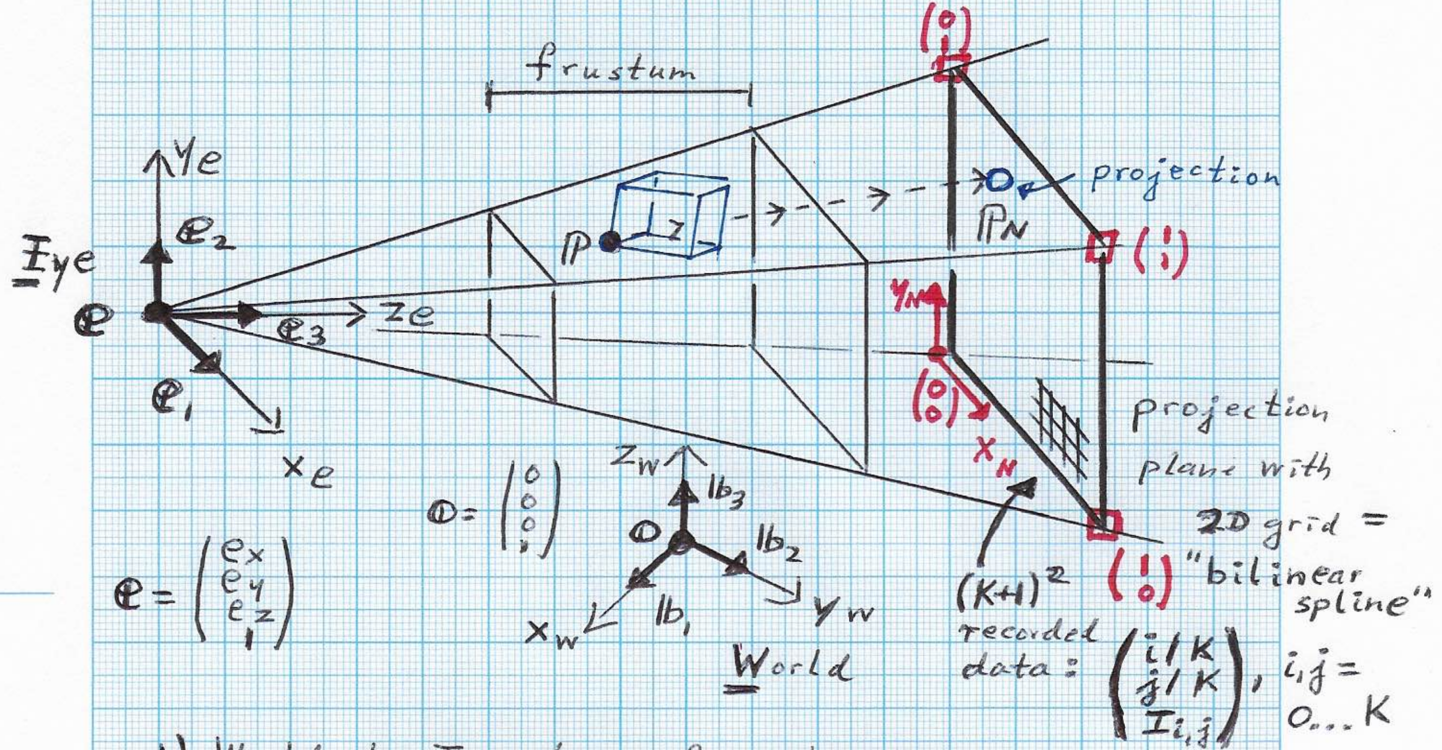
RECONSTRUCTION – 3D PERSPECTIVE PROJECTION

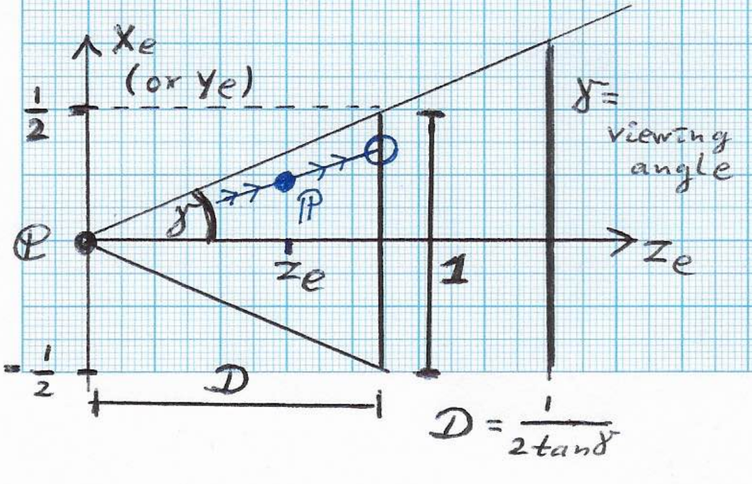
(NOTATION & REVIEW)



1) World-to-Eye transformation:

$$\underline{\underline{\mathbb{P}_E}} = \left[\begin{array}{c|c} \vdash \mathbb{e}_1 \dashv & 0 \\ \vdash \mathbb{e}_2 \dashv & 0 \\ \vdash \mathbb{e}_3 \dashv & 0 \\ \hline 0 \quad 0 \quad 0 & 1 \end{array}\right] \cdot \left[\begin{array}{ccc|c} 1 & 0 & 0 & -e_x \\ 0 & 1 & 0 & -e_y \\ 0 & 0 & 1 & -e_z \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \cdot \mathbb{P}_W ,$$

$$\mathbb{P}_W = \begin{pmatrix} x_w \\ y_w \\ z_w \\ 1 \end{pmatrix}, \quad \mathbb{P}_E = \begin{pmatrix} x_e \\ y_e \\ z_e \\ 1 \end{pmatrix}$$



2) Eye-to-Square

(map to normalized projection area):

$$\underline{\underline{\mathbb{P}_N}} = \frac{1}{2\tan\gamma} \begin{pmatrix} x_e/z_e \\ y_e/z_e \end{pmatrix} + \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix},$$

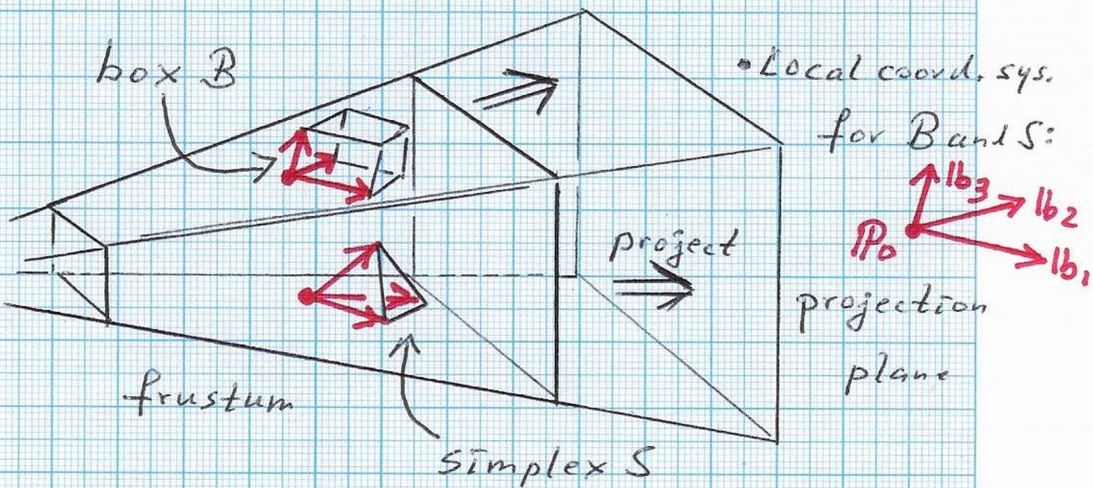
$$z_e \neq 0, \ \tan\gamma \neq 0$$

■ RECONSTRUCTION – TOWARDS ANALYTICAL METHODS: EXACT PROJECTIONS OF BOXES & SIMPLICES

{→ Related literature:

- Prautzsch, Boehm & Paluszny: "Bézier and B-spline Techniques" (chapters on BOX SPLINES and SIMPLEX SPLINES)
- de Boor, Höllig & Riemenschneider: "Box Splines"
- Papers by Alireza Entezari et al.:
  - "Box Spline Projection in Non-parallel Geometry"
  - "Box Spline Reconstruction...", "Linear and Cubic Box Splines..."
- Papers by Klaus Müller et al. and Torsten Möller et al.}

• Goal: Exact analytical definition of projection ("shadow" / "X-ray image") of box & simplex



⇒ Points inside a box or simplex solid can be expressed analytically:

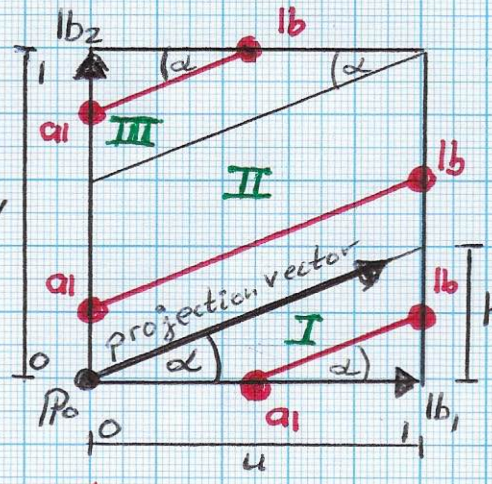
$$\mathbb{P} = \mathbb{p}_0 + u_1 \mathbb{b}_1 + u_2 \mathbb{b}_2 + u_3 \mathbb{b}_3$$

⇒ analytical def. of shadows of box B & simplex S

Reconstruction: Towards Analytical Methods – Cont'd.

• **Example:** 2D parallel projection of a box

$\alpha \in (0°, 45°)$:



(Similar for $\alpha \in (45°, 90°)$)

• Determine lengths of segments in regions I, II, III:

$$\underline{\tan\alpha = t\alpha = h}$$

• I) $\mathbb{a}_1 = \begin{pmatrix} u \\ 0 \end{pmatrix}$, $\mathbb{b} = \begin{pmatrix} 1 \\ v \end{pmatrix}$

$$\tan\alpha = \frac{v}{1-u} = t\alpha$$

$$\Rightarrow v = (1-u)\,t\alpha,$$
$$u = 1 - v/t\alpha$$
$$\Rightarrow \mathbb{a}_1 = \begin{pmatrix} 1 - v/t\alpha \\ 0 \end{pmatrix}$$

$$\Rightarrow \|\mathbb{b} - \mathbb{a}_1\| = L(v) = \underline{v\sqrt{1 + 1/t^2\alpha}}$$

• II) $\mathbb{a}_1 = \begin{pmatrix} 0 \\ v \end{pmatrix}$, $\mathbb{b} = \begin{pmatrix} 1 \\ v+h \end{pmatrix} = \begin{pmatrix} 1 \\ v + t\alpha \end{pmatrix}$

$$\Rightarrow \underline{L(v) = \sqrt{1 + t^2\alpha}}$$

• III) $\mathbb{a}_1 = \begin{pmatrix} 0 \\ v \end{pmatrix}$, $\mathbb{b} = \begin{pmatrix} u \\ 1 \end{pmatrix}$

$$\tan\alpha = \frac{1-v}{u} = t\alpha$$

$$\Rightarrow v = 1 - u\,t\alpha,$$
$$u = (1-v)/t\alpha$$
$$\Rightarrow \mathbb{b} = \begin{pmatrix} (1-v)/t\alpha \\ 1 \end{pmatrix}$$

$$\Rightarrow \|\mathbb{b} - \mathbb{a}_1\| = L(v) = \underline{(1-v)\sqrt{1 + 1/t^2\alpha}}$$

• **Summary:**

I) $L(v) = v\sqrt{1 + 1/t^2\alpha}$
($v = 0 \ldots h$ for $\mathbb{b}$)

II) $L(v) = \sqrt{1 + t^2\alpha}$
($v = h \ldots (1-h)$ for $\mathbb{a}_1$)

III) $L(v) = (1-v)\sqrt{1 + 1/t^2\alpha}$
($v = (1-h) \ldots 1$ for $\mathbb{a}_1$)

• **Summary (alternative: $L = L(u)$):**

I) $L(u) = (1-u)\sqrt{1 + t^2\alpha}$
($u = 0 \ldots 1$ for $\mathbb{a}_1$)

II) $L(u) = \sqrt{1 + t^2\alpha}$

III) $L(u) = u\sqrt{1 + t^2\alpha}$
($u = 0 \ldots 1$ for $\mathbb{b}$)

~BH