Bernd Hamann

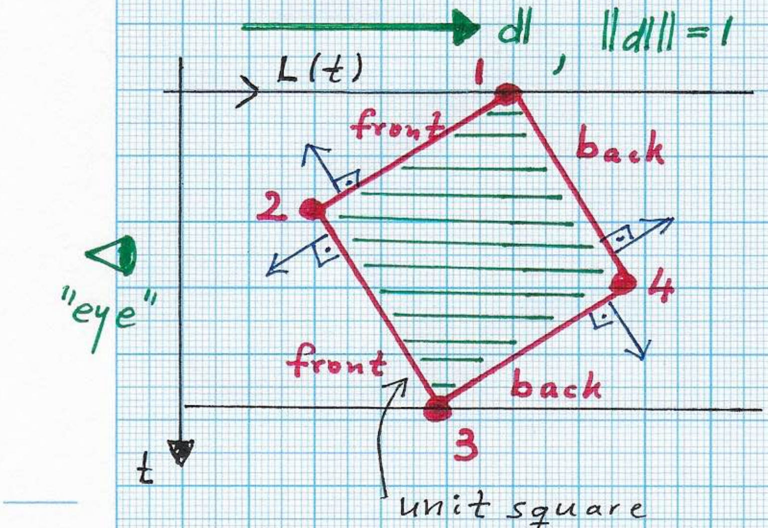
5/29/2020

Stratovan

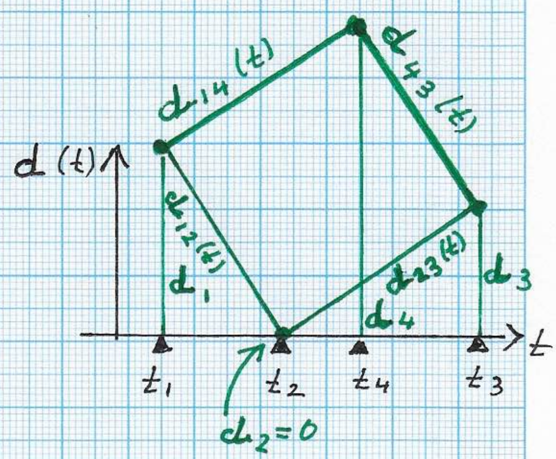
RECONSTRUCTION: ANALYTICAL PROJECTION OF SQUARES/CUBES

1) 2D Case - Square

i) Parallel Projection

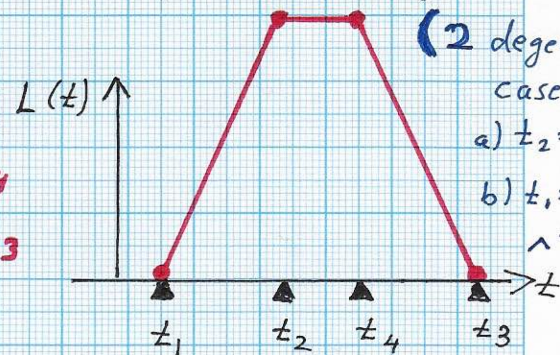


• distance functions:



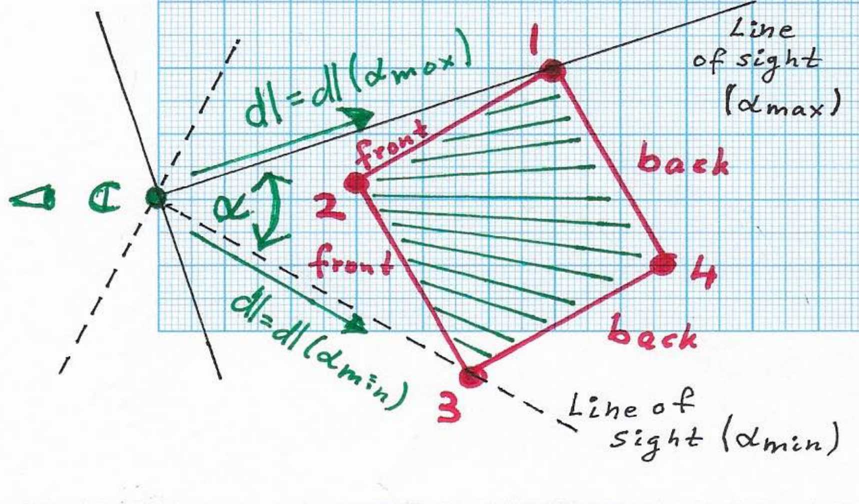
• ray segment Length function:

$$\underline{\underline{L(t)}} = \begin{cases} d_{14}(t) - d_{12}(t), & t_1 \dots t_2 \\ d_{14}(t) - d_{23}(t), & t_2 \dots t_4 \\ d_{43}(t) - d_{23}(t), & t_4 \dots t_3 \end{cases}$$

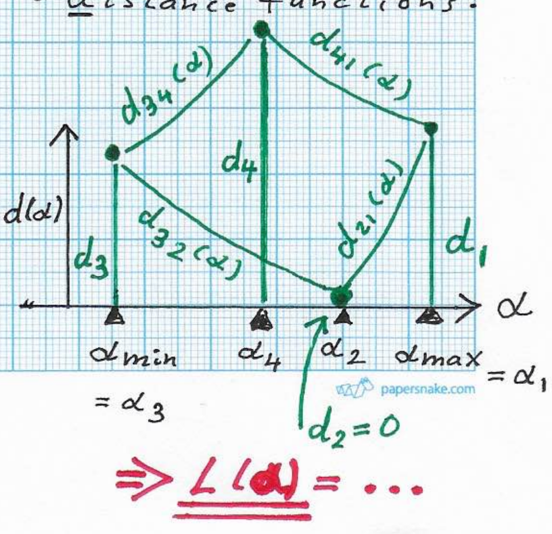


(2 degenerate cases:
a) $t_2 = t_4$
b) $t_1 = t_2 \wedge t_4 = t_3$)

ii) Perspective Projection



• distance functions:



$\Rightarrow \underline{\underline{L(\alpha)}} = \dots$

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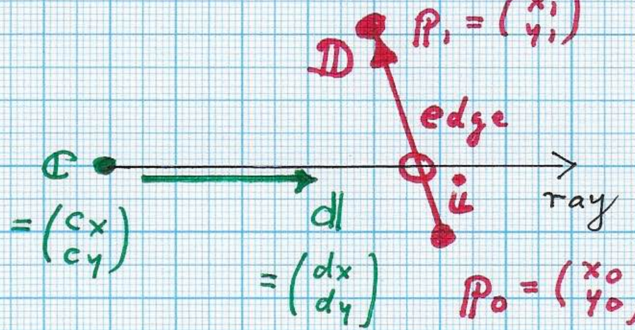
▪ RECONSTRUCTION: ANALYTICAL PROJECTION – Cont'd.

1) ii) Square, perspective projection:

→ Intersection of IMPLICIT square edge and PARAMETRIC ray:



• ray equation:

$$\mathbb{X}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \mathbb{C} + t\,\mathbb{d}$$

• edge equation:

$$\mathbb{D} = \mathbb{P}_1 - \mathbb{P}_0 = \begin{pmatrix} D_x \\ D_y \end{pmatrix}$$

⇒ normal to $\mathbb{D}$ is

$$\mathbb{D}^\times = \begin{pmatrix} -D_y \\ D_x \end{pmatrix}$$

⇒ point $\mathbb{X} = \begin{pmatrix} x \\ y \end{pmatrix}$ on (line defined by) edge

$$\Leftrightarrow \quad \mathbb{D}^\times \bullet (\mathbb{X} - \mathbb{P}_0) = 0$$

• RAY:

$$\mathbb{X} = \begin{pmatrix} c_x \\ c_y \end{pmatrix} + t \begin{pmatrix} d_x \\ d_y \end{pmatrix} \quad \Leftarrow$$

• EDGE:

$$\begin{pmatrix} -D_y \\ D_x \end{pmatrix} \bullet \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} = 0$$

• Insert ray into edge:

$$\begin{pmatrix} -D_y \\ D_x \end{pmatrix} \bullet \begin{pmatrix} c_x + t d_x - x_0 \\ c_y + t d_y - y_0 \end{pmatrix} = 0$$

$$\hookrightarrow \quad -D_y c_x - t D_y d_x + D_y x_0 + D_x c_y + t D_x d_y - D_x y_0 = 0$$

$$\hookrightarrow \quad t \cdot (D_x d_y - D_y d_x) = D_y c_x - D_x c_y + D_x y_0 - D_y x_0$$

$$t \cdot (\mathbb{D}^\times \bullet \mathbb{d}) = -\mathbb{D}^\times \bullet \mathbb{C} + \mathbb{D}^\times \bullet \mathbb{X}_0$$

$$\hookrightarrow \quad t \cdot (\mathbb{D}^\times \bullet \mathbb{d}) = \mathbb{D}^\times \bullet (\mathbb{X}_0 - \mathbb{C})$$

$$\hookrightarrow \quad t = \frac{\mathbb{D}^\times \bullet (\mathbb{X}_0 - \mathbb{C})}{\mathbb{D}^\times \bullet \mathbb{d}}, \quad \mathbb{D}^\times \bullet \mathbb{d} \neq 0$$

$$\Rightarrow \quad \mathbb{i} = \mathbb{C} + t\,\mathbb{d}$$ (special case: $\mathbb{C} = \mathbb{0}$)

• **Note:** $\mathbb{i}$ is on edge $\overline{\mathbb{P}_0\mathbb{P}_1} \Leftrightarrow (\mathbb{i} - \mathbb{P}_0) = u \cdot \mathbb{D} \wedge u \in [0,1]$.

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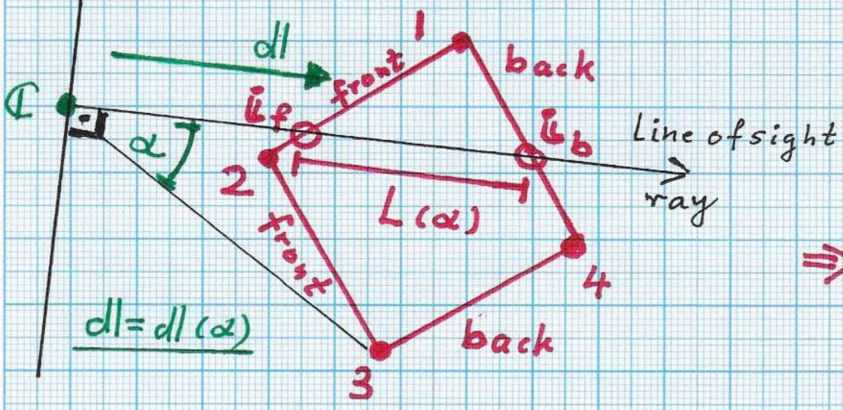
RECONSTRUCTION: ANALYTICAL PROJECTION – Cont'd.

1) ii) Square, perspective projection:



$\mathbf{u}_b = \mathbf{u}_{back}(\alpha)$

$\mathbf{u}_f = \mathbf{u}_{front}(\alpha)$

$$\Rightarrow \underline{\underline{L(\alpha) =}}$$

$$d_{41}(\alpha) - d_{21}(\alpha)$$

$$= \underline{\underline{\|\mathbf{u}_b - \mathbf{u}_f\|}}$$

• **Note:** Alternatively, one can represent a square edge parametrically and a ray implicitly.
⇒ The resulting equations are equivalent.

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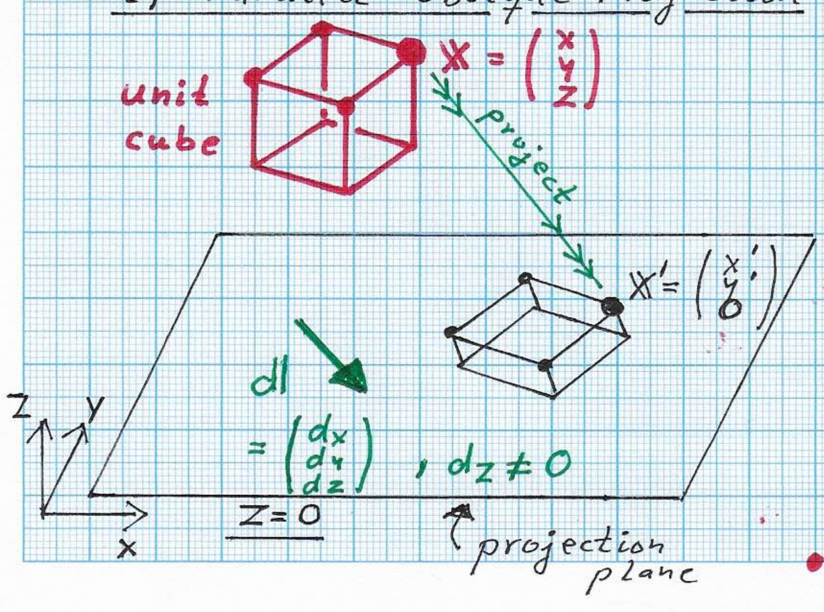
2) 3D Case – Cube

i) Parallel-oblique Projection



• Review: Oblique projection (plane $z = 0$, direction $\mathbf{dl}$):

$$\mathbf{x}' = \mathbf{x} + t\,\mathbf{dl}$$

$(z' = 0 = z + t\,d_z$
$\hookrightarrow t = -z/d_z)$

$$\underline{\underline{\mathbf{x}' = \begin{pmatrix} x' \\ y' \\ 0 \end{pmatrix} = \begin{pmatrix} x - \frac{d_x}{d_z} z \\ y - \frac{d_y}{d_z} z \\ 0 \end{pmatrix}}}$$

• Each cube face defines a linear distance function with the face's projection as domain.

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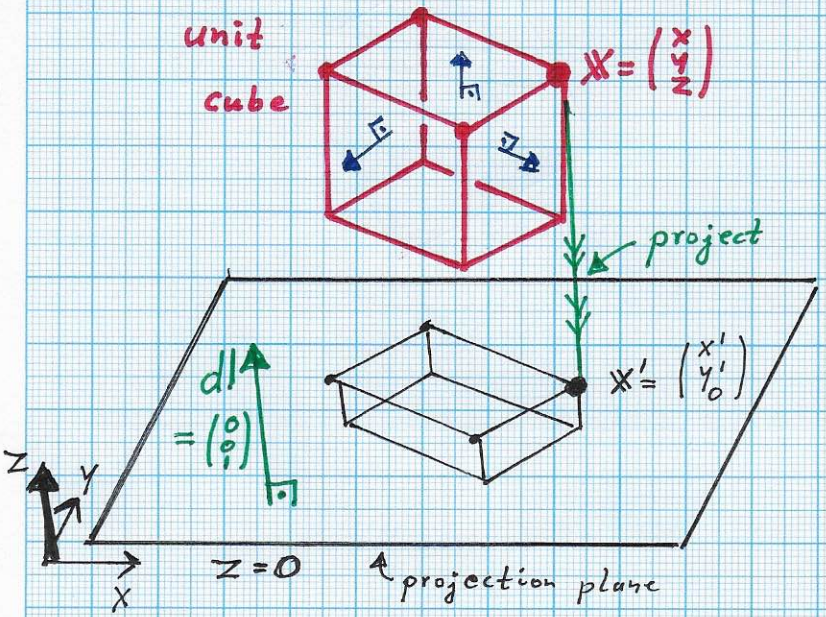
RECONSTRUCTION: ANALYTICAL PROJECTION - Cont'd.

... 3D Case - Cube

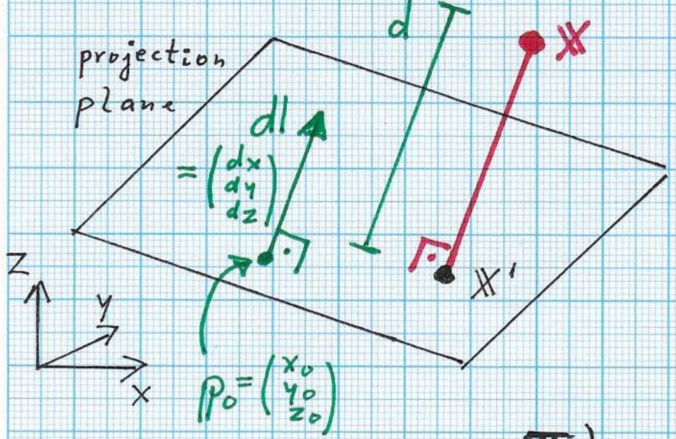
i) Parallel - ORTHOGRAPHIC Projection only!



- Review: Orthographic projection (plane=0):

$$\mathbb{X}' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

- General orthographic projection (projection plane defined by unit normal $\mathbb{d} = \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$ and point $\mathbb{p}_0 = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$ in the projection plane)



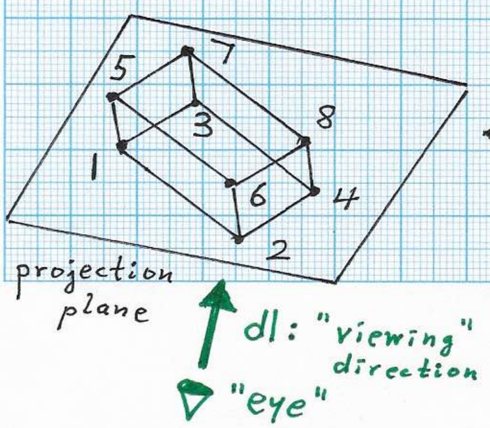
$\Longleftrightarrow$ I) distance d between point $\mathbb{X}$ and $\mathbb{X}'$:

$$d = \mathbb{d} \cdot (\mathbb{X} - \mathbb{p}_0),$$

$$\|\mathbb{d}\| = 1$$

II) projected point $\mathbb{X}'$ in plane:

$$\mathbb{X}' = \mathbb{X} - d \cdot \mathbb{d}$$



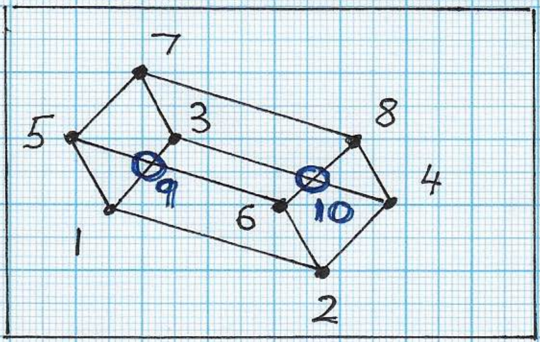
$\Longleftrightarrow$

- points $1, 2, \ldots, 8$ = cube corners projected in plane
- (except in degenerate cases) three quadrilaterals represent BACK faces, three quadrilaterals represent FRONT faces.
- Example: FACES 5678, 1357, 3478, 1234, 1256, 2468

RECONSTRUCTION: ANALYTICAL PROJECTION – Cont'd.

... orthographic projection of unit cube ...

→ projection plane, top-down view:



- The six unit square faces of the unit cube project to six overlapping quadrilaterals in the projection plane.

- Each point in $\{1, 2, \ldots, 8\}$ has an associated distance value $d_1, d_2, \ldots, d_8$ (= distances between original cube corners and projection plane).
- SINCE CUBE FACES ARE PLANAR, EACH QUAD IN THE PROJECTION PLANE DEFINES A LINEAR DISTANCE FUNCTION FOR THE QUAD.

- THE LENGTHS OF RAYS PASSING THROUGH THE CUBE HAVE SEGMENT LENGTHS (INSIDE THE CUBE) THAT ARE (ABSOLUTE) DIFFERENCES OF QUAD-SPECIFIC LINEAR DISTANCE FUNCTIONS.

{• It is necessary to compute and consider points 9, 10.}

- The **L-function** defined over a domain given by the union area of all six quad areas is defined as a **piecewise linear function** as follows:

RECONSTRUCTION: ANALYTICAL PROJECTION – Cont'd.

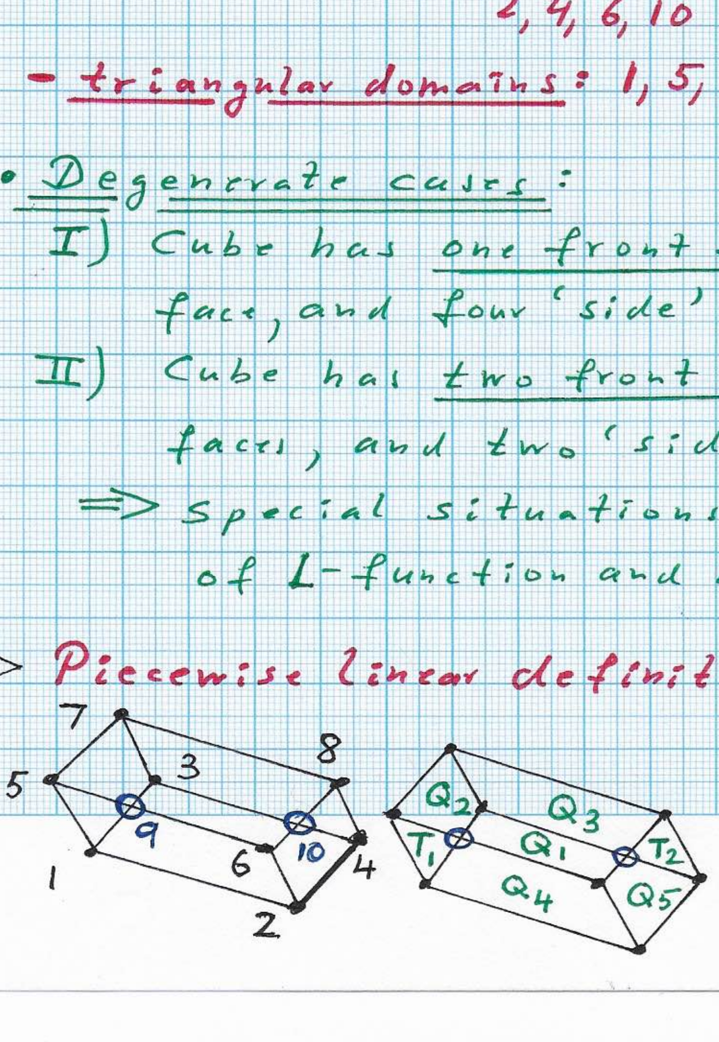
... the L-function for a cube ...

→ See image on previous page... (example)
BACK faces: 5, 6, 7, 8; 1, 2, 5, 6; 2, 4, 6, 8
FRONT faces: 1, 2, 3, 4; 1, 3, 5, 7; 3, 4, 7, 8
(INTERSECTIONS of quads: 9, 10)

⇒ L-function has SEVEN linear pieces, defined for the following quadrilateral and triangular sub-domains:
- quad domains: 3, 6, 9, 10; 3, 5, 7, 9; 1, 2, 6, 9; 3, 7, 8, 10; 2, 4, 6, 10
- triangular domains: 1, 5, 9; 4, 8, 10

{ • Degenerate cases:
I) Cube has one front face, one back face, and four 'side' faces.
II) Cube has two front faces, two back faces, and two 'side' faces.
⇒ special situations for domain of L-function and L-fct definition }

⇒ Piecewise linear definition of L-function:



- left: projected points
- right: resulting quads $(Q_1, \ldots, Q_5)$ and triangles (T_1, T_2) ⇒ sub-domains of L-function

RECONSTRUCTION: ANALYTICAL PROJECTION – Cont'd.

... the **L-function** ...

- $d_{i,j,k,l}$ = (linear) distance function associated with cube corners i, j, k, l / with face with corners i, j, k, l

- L-function is defined piecewise, with one linear piece for each quadrilateral and triangular sub-domain; above example:

$$L = \begin{cases} d_{5,6,7,8} - d_{1,2,3,4} & Q_1 \\ d_{5,6,7,8} - d_{1,3,5,7} & Q_2 \\ d_{5,6,7,8} - d_{3,4,7,8} & Q_3 \\ d_{1,2,5,6} - d_{1,2,3,4} & Q_4 \\ d_{2,4,6,8} - d_{1,2,3,4} & Q_5 \\ d_{1,2,5,6} - d_{1,3,5,7} & T_1 \\ d_{2,4,6,8} - d_{3,4,7,8} & T_2 \end{cases}$$

- The differences of d-functions are non-negative.

∴ BH