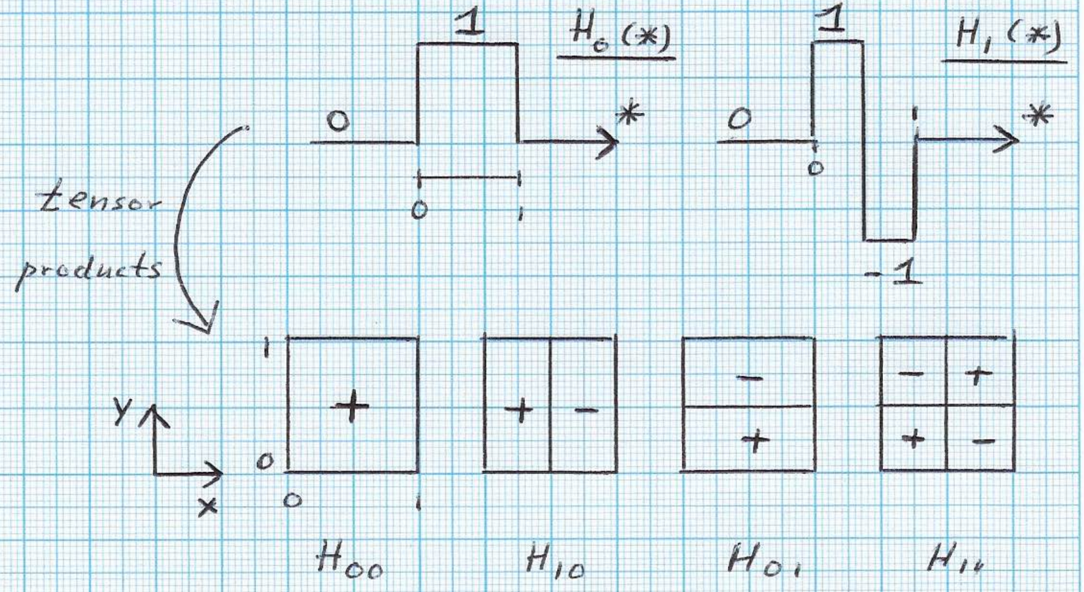
Stratovan

■ WAVELET RECONSTRUCTION – DIRECT RECONSTRUCTION OF AN IMAGE USING HAAR WAVELETS

- **Goal:**
  - Use orthogonal and normalized Haar basis functions for image reconstruction and representation
  - Use the scale-specific coefficients of the wavelet representation to perform scale-specific analysis and classification
  - Expected efficiency gains include data storage and processing time.

- **Example:** – 2D tensor product Haar wavelet representation for 2×2 final resolution



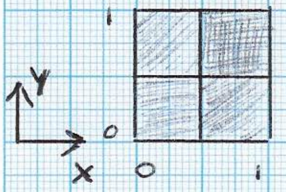
where $H_{ij} = H_{i,j}(x,y) = H_i(x)\,H_j(y)$

⇒ • Functions H_{ij} are pairwise orthogonal.

• Normalize H_{ij}: $\|H_{ij}\| = 1$, i.e., $\sqrt{\int (H_{ij})^2} = 1$.

WAVELET RECONSTRUCTION – Cont'd.

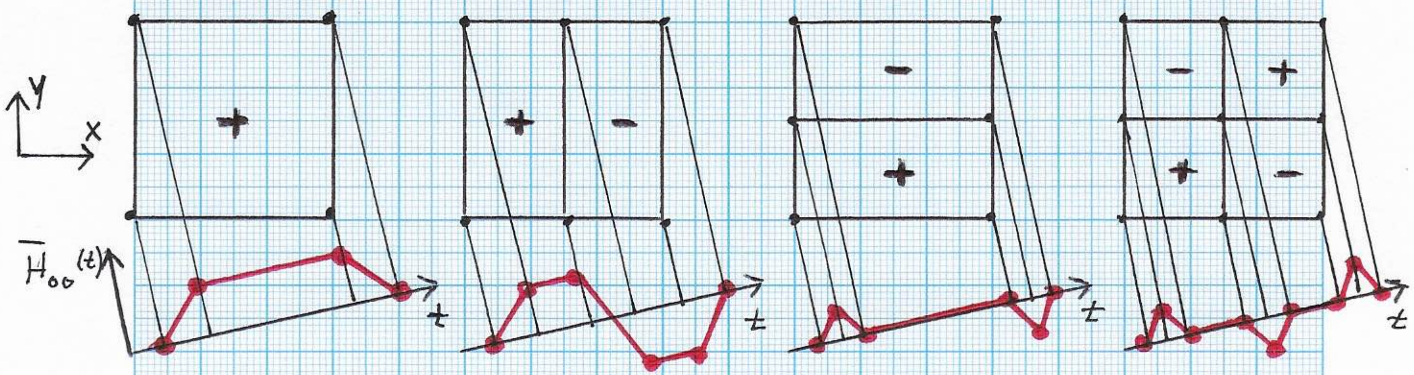
- Represent the 2×2 image to be reconstructed in 2D tensor product Haar wavelet basis:



$$\Leftarrow \quad I(x,y) = \sum_{i=0}^{1} \sum_{j=0}^{1} c_{ij} \, H_{ij}(x,y)$$

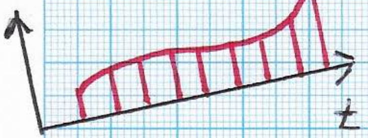
- advantage: $|c_{ij}|$ defines "importance" of basis function H_{ij}.

- Must determine the x-rays/shadows/projections of basis functions H_{ij} on 1D projection line!



$\overline{H}_{00}(t)$ $\quad$ $\overline{H}_{10}(t)$ $\quad$ $\overline{H}_{01}(t)$ $\quad$ $\overline{H}_{11}(t)$

- Compute a BEST APPROXIMATION of the projection of an image I when projected onto t-line:



$\overline{I}(t)$

(= sampled, discrete piecewise const. function)

$$\overline{I}_a(t) = \sum_{i=0}^{1} \sum_{j=0}^{1} \overline{c}_{ij} \, \overline{H}_{ij}(t)$$

such that $\| \overline{I}_a(t) - \overline{I}(t) \| \rightarrow \min$

($\overline{I}_a(t)$ denotes least squares approximation of $\overline{I}(t)$, computed via solving the NORMAL EQUATIONS.)

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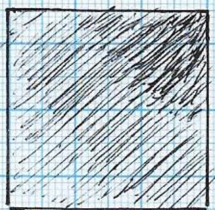
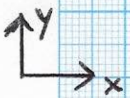
WAVELET RECONSTRUCTION - Cont'd.

- Relationships between representations and computations in "original 2D image space" and "measured 1D projection space"

0) CONTINUOUS (not-yet-discretized) original image $\hat{I}(x,y)$

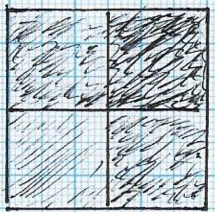
$\hat{I}(x,y)$





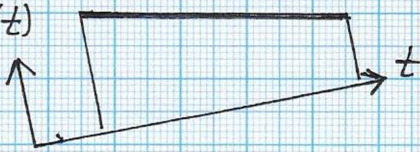
1) MESH-BASED / GRIDDED image $I(x,y)$ (assumed to be a least-squares best approximation of $\hat{I}(x,y)$ in Haar wavelet basis)

$I(x,y)$



2) CONTINUOUS projection of $\hat{I}(x,y)$: $\hat{I}(t)$ (not known in physical setting)

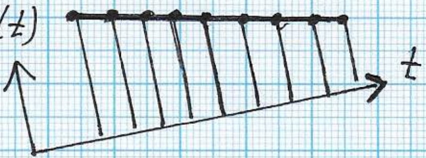
$\hat{I}(t)$



3) SAMPLED projection of $\hat{I}(x,y)$: $\overline{\hat{I}}(t)$ (typical scanner measurements)

4) SAMPLED projection of $I(x,y)$: $\overline{I}(t)$ (can also be interpreted as $\overline{\hat{I}}$)

$\overline{\hat{I}}(t)$

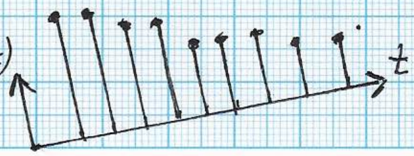


5) 2D Haar basis functions: $H_{ij}(x,y)$

6) 1D projections of Haar basis fcts: $\overline{H}_{ij}(t)$

- Representations:

$\overline{I}(t)$



$$I(x,y) = \sum_{i=0}^{1} \sum_{j=0}^{1} c_{ij} H_{ij}(x,y)$$

$$\overline{\hat{I}}_a(t) = \sum_{i=0}^{1} \sum_{j=0}^{1} \overline{c}_{ij} \overline{H}_{ij}(t)$$

= BEST APPROX. of $\overline{\hat{I}}(t)$ using projected Haar basis fcts. on t-line

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WAVELET RECONSTRUCTION – Cont'd.

- Using the "PROJECTION operator" $\mathcal{P}$:

$$\mathcal{P}\left(I(x,y)\right) = \mathcal{P}\left(\sum_{i=0}^{1}\sum_{j=0}^{1} c_{ij}\, H_{ij}(x,y)\right)$$

$$= \sum_{i=0}^{1}\sum_{j=0}^{1} c_{ij}\, \mathcal{P}\left(H_{ij}(x,y)\right)$$

$$= \sum_{i=0}^{1}\sum_{j=0}^{1} c_{ij}\, \overline{H_{ij}}(t)$$

$= \overline{c_{ij}}$!

! "Projection of an Image given as a sum of wavelet basis functions

= sum of projections of wavelet basis functions"

$\Rightarrow$ Principle for computation of wavelet reconstruction:

- COMPUTE BEST APPROXIMATION OF RECORDED DATA / FUNCTION OVER THE t-LINE USING THE PROJECTED HAAR WAVELET FUNCTIONS AS BASIS FUNCTIONS (OVER THE t-LINE)
  – SOLVING THE NORMAL EQUATIONS RESULTING FROM LEAST SQUARES
- USE THE COMPUTED VALUES $\overline{c_{ij}} = c_{ij}$ TO DEFINE A HAAR WAVLET REPRESENTATION (APPROXIMATION) OF $\hat{I}(x,y)$.

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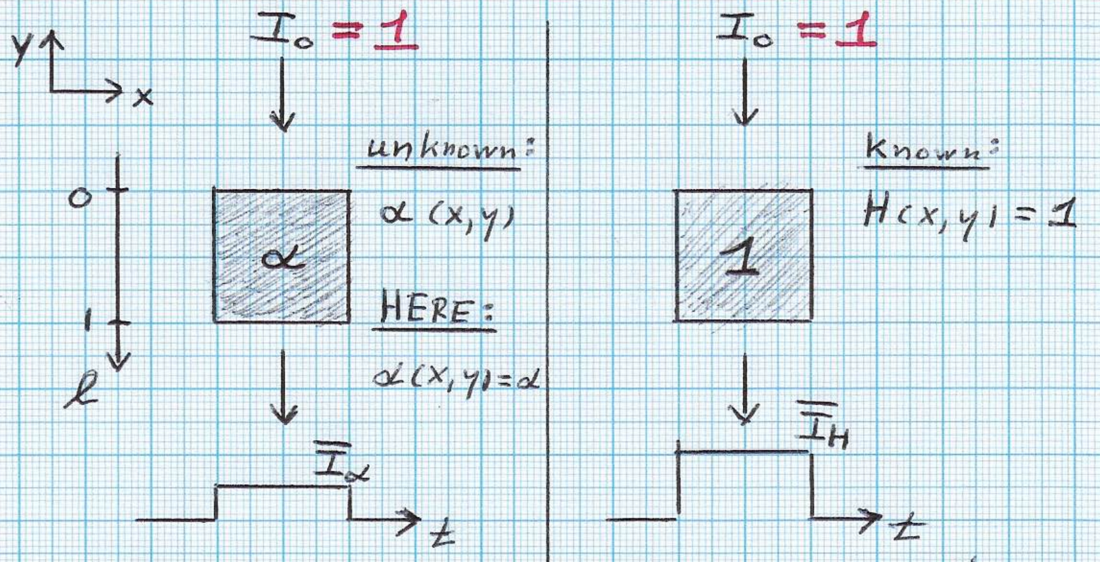
WAVELET RECONSTRUCTION – Cont'd.

- Determining wavelet representation (of reconstructed image) by expressing recorded/measured projection of object via expansion defined by projections of wavelet basis functions...

i) Example: ONE wavelet basis function:



$$\bar{I}_\alpha(t) = I_0 e^{-\int_0^1 \alpha \, dl} \qquad \bar{I}_H(t) = I_0 e^{-\int_0^1 1 \, dl}$$

measured $\quad = I_0 e^{-\alpha}$ $\qquad$ known $\quad = I_0 e^{-1}$

$$\Rightarrow \alpha = -\ln(\bar{I}_\alpha / I_0) = -\ln(w_\alpha) \qquad \Rightarrow 1 = -\ln(\bar{I}_H / I_0) = -\ln(w_H)$$

- Wavelet representation of $\alpha(x,y)$:

$\Rightarrow$ Determine coefficient c: $-\ln(w_\alpha) = c(-\ln(w_H))$

$$\hookrightarrow c = \ln(\bar{I}_\alpha / I_0) / \ln(\bar{I}_H / I_0)$$

$$= \frac{\ln(\bar{I}_\alpha) - \ln(I_0)}{\ln(\bar{I}_H) - \ln(I_0)} \overset{I_0 = 1}{=} \ln(e^{-\alpha}) / \ln(e^{-1}) = \alpha$$

$$\boxed{\alpha(x,y) = \frac{\ln e^{-\alpha}}{\ln e^{-1}} H(x,y)}$$

$= \alpha$ ✓

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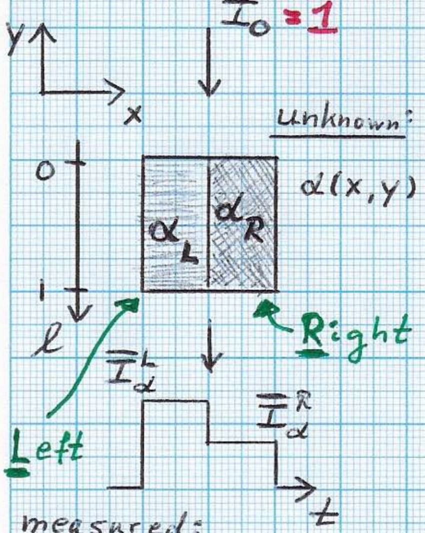
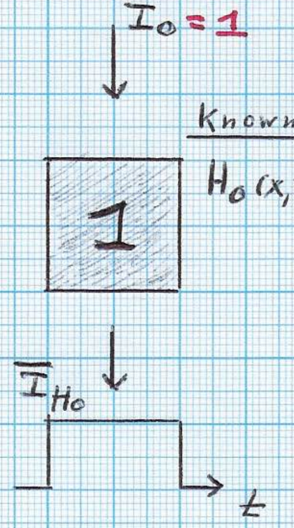
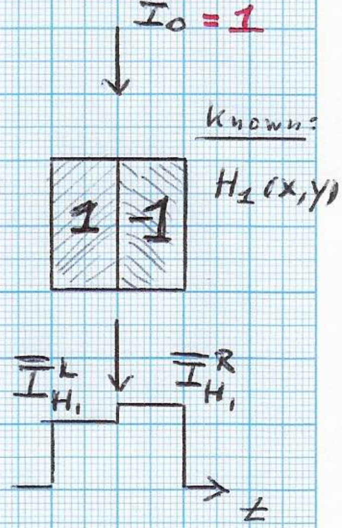
■ WAVELET RECONSTRUCTION – Cont'd.

ii) Example: TWO wavelet basis functions:







measured:

$$\bar{I}^L_\alpha(t) = I_0 e^{-\alpha_L} \Rightarrow \alpha_L = -\ln\left(\frac{\bar{I}^L_\alpha}{I_0}\right)$$

$$\bar{I}^R_\alpha(t) = I_0 e^{-\alpha_R} \Rightarrow \alpha_R = -\ln\left(\frac{\bar{I}^R_\alpha}{I_0}\right)$$

Known:

$$\bar{I}_{H_0} = I_0 e^{-1} \Rightarrow 1 = -\ln(\bar{I}_{H_0}/I_0)$$

$$\bar{I}^L_{H_1} = I_0 e^{-1} \Rightarrow 1 = -\ln(\bar{I}^L_{H_1}/I_0)$$

$$\bar{I}^R_{H_1} = I_0 e^{1} \Rightarrow -1 = -\ln(\bar{I}^R_{H_1}/I_0)$$

$$\Rightarrow \boxed{\begin{aligned} \alpha_L &= -\ln(w^L_\alpha) \\ \alpha_R &= -\ln(w^R_\alpha) \end{aligned}} \qquad \Rightarrow \boxed{\begin{aligned} 1 &= -\ln(w_{H_0}) \\ 1 &= -\ln(w^L_{H_1}) \\ -1 &= -\ln(w^R_{H_1}) \end{aligned}}$$

$\Rightarrow$ Determine coefficients c_0 and c_1:

Linear equation system:

$$\left.\begin{aligned} \text{Left:} \quad 1\,c_0 + 1\,c_1 &= \alpha_L \\ \text{Right:} \quad 1\,c_0 - 1\,c_1 &= \alpha_R \end{aligned}\right\} \Rightarrow \ldots \quad \begin{aligned} c_0 &= (\alpha_L + \alpha_R)/2 = -\tfrac{1}{2}\left(\ln(w^L_\alpha) + \ln(w^R_\alpha)\right) \\ c_1 &= (\alpha_L - \alpha_R)/2 = -\tfrac{1}{2}\left(\ln(w^L_\alpha) - \ln(w^R_\alpha)\right) \end{aligned}$$

- **Wavelet repr. of $\alpha(x,y)$:** $\alpha(x,y) = \sum_{i=0}^{1} c_i H_i(x,y)$ ✓