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On Material/Object Classification, BASIS PURSUIT, Linear Systems & GEOMETRICAL MEANING...

→ **Problem:** Given K points $\mathbb{p}_i$ (= positional vectors, "ATOMS"), express any point $\mathbb{p}$ as an "optimal" linear combination using a subset of $\{\mathbb{p}_i\}_{i=1}^{K}$, where $\mathbb{p}_i$ and $\mathbb{p}$ are points in $\mathbb{R}^N$ **(K>N)**, i.e.,

$$\mathbb{p} = \sum_{j \in I} c_j \mathbb{p}_j \quad , \; I \subseteq \{1, \ldots, K\}.$$

- **Notes:** (i) The N-dimensional atoms represent N-dimensional histograms (= binned histograms using N bins) of an object's properties.

  (ii) The atoms $\mathbb{p}_i$ are NORMALIZED, i.e., $\|\mathbb{p}_i\| = 1$, meaning that all atoms lie on a hyper-sphere of radius 1 embedded in N-dimensional space.

  (iii) In addition to (ii), all coordinates/components of every atom $\mathbb{p}_i$ are NON-NEGATIVE.

→ **Goal:** Devise a GEOMETRICAL method for determining the desired subset of atoms $\{\mathbb{p}_j\}$, i.e., determine the best N atoms to represent $\mathbb{p}$ as "the ideal" linear combination using these N atoms.

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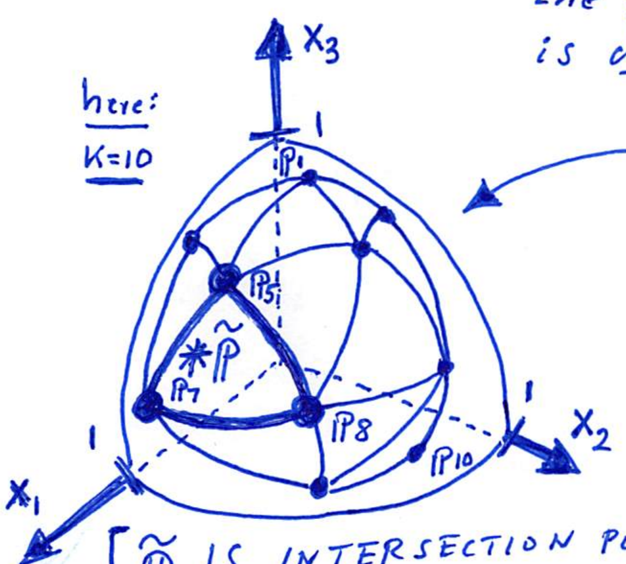
... Geometrical Meaning of Basis Pursuit (Cont'd)...

[REF: Patrick HUGGINS and Steven Zucker, Greedy Basis Pursuit, IEEE Trans. on Signal Proc.. Vol. 55, No. 7, pp. 3760 ff.]

• Example: – $N=3$, i.e., points $\mathbb{p}_i$, $i=1\ldots K$, lie on the octant of the unit sphere in 3D space where all points' coordinates are non-negative.

– Thus: The points $\mathbb{p}_i$ lie on the sphere; an arbitrary point $\mathbb{p}$ is given (not necessarily lying on the sphere); the best subset – consisting of 3 atoms – is determined to represent $\mathbb{p}$.



GEOMETRY / COMPLEX:

→ In 3D space, the ten atoms $\mathbb{p}_i$ define a spherical Delaunay triangulation / complex.

→ Point $\tilde{\mathbb{p}}$ lies in one of the Delaunay triangles (or on an edge...).

[$\tilde{\mathbb{p}}$ IS INTERSECTION POINT OF LINE THROUGH ORIGIN AND $\mathbb{p}$ WITH THIS SPHERICAL OCTANT.]

→ ONE MUST DETERMINE THE (SPHERICAL) TRIANGLE CONTAINING $\tilde{\mathbb{p}}$.

[Here, that triangle has vertices $\mathbb{p}_5, \mathbb{p}_7, \mathbb{p}_8$.]

→ OPTIMAL REPRESENTATION: $\boxed{\mathbb{p} = c_5\mathbb{p}_5 + c_7\mathbb{p}_7 + c_8\mathbb{p}_8 .}$

→ NEEDED: SIMPLE, ROBUST, EFFICIENT ALGORITHM FOR FINDING THAT DELAUNAY SIMPLEX – here: triangle – ON HYPER-SPHERE THAT CONTAIN THE POINT $\tilde{\mathbb{p}}$. [Computational Geometry???]

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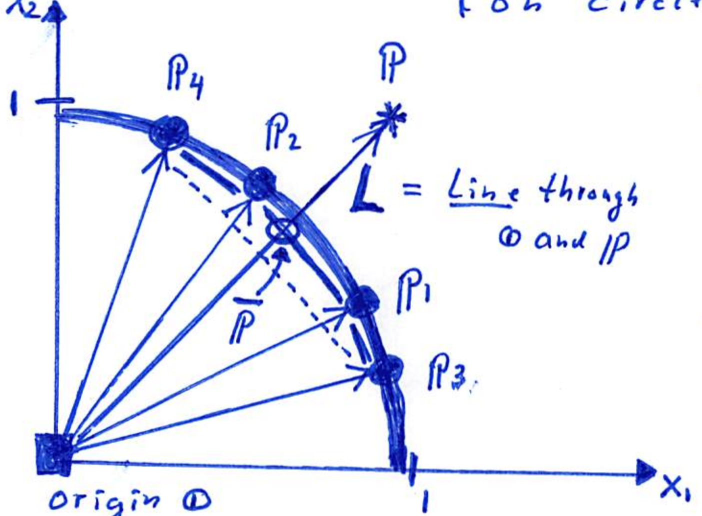
... Geometrical Meaning of Basis Pursuit (Cont'd)...

AND A CONSTRUCTIVE ALGORITHM USING CONCEPTS FROM COMPUTATIONAL GEOMETRY...

- **Example**: – $N=2$, i.e., points $\mathbb{P}_i$ on circular arc in positive quadrant of $x_1 x_2$-system (on circle with radius 1)



– $K=4 \Rightarrow$ must determine 2 points from $\{\mathbb{P}_i\}_{i=1}^{4}$ that define best representation of $\mathbb{P}$.

– Here: The 2 "best atoms" are $\mathbb{P}_1$ and $\mathbb{P}_2$.

– THE BOUNDARY OF THE CONVEX HULL OF POINTS $\mathbb{P}_1, \ldots, \mathbb{P}_4$ ARE THE EDGES $\overline{\mathbb{P}_3\mathbb{P}_1}$, $\overline{\mathbb{P}_1\mathbb{P}_2}$, $\overline{\mathbb{P}_2\mathbb{P}_4}$, $\overline{\mathbb{P}_4\mathbb{P}_3}$. !

– THE LINE $\overline{O\mathbb{P}}$ HAS 2 INTERSECTIONS WITH THE BOUNDARY OF THE CONVEX HULL – ONE BEING THE POINT $\overline{\mathbb{P}}$. !

– POINT $\overline{\mathbb{P}}$ IS INSIDE THE EDGE $\overline{\mathbb{P}_1\mathbb{P}_2}$ (= the edge that is NOT the edge connecting the 'extremal points' $\mathbb{P}_3$ and $\mathbb{P}_4$). !

– THUS: OPTIMAL REP. OF $\mathbb{P}$: $\boxed{\mathbb{P} = c_1\mathbb{P}_1 + c_2\mathbb{P}_2}$

!!! • **NEEDED**: ALGORITHM DETERMINING THE SIMPLICIAL FACET OF THE BOUNDARY OF THE CONVEX HULL OF $\{\mathbb{P}_i\}$ THAT IS INTERSECTING WITH **L** IN POINT $\overline{\mathbb{P}}$.

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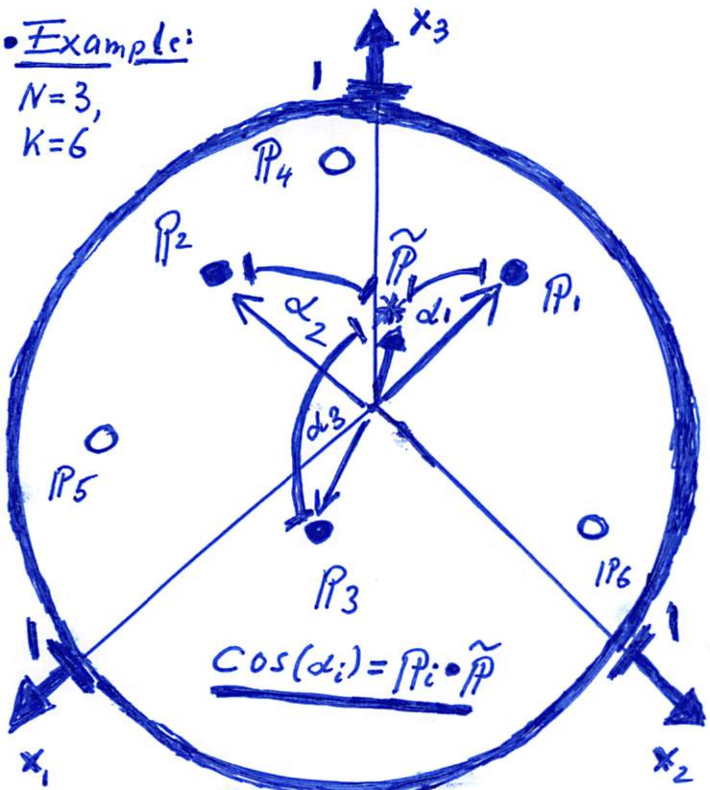
... CONSTRUCTIVE ALGORITHM for Basis Pursuit

- Using a Simple, Intuitive Approach

• Example:
$N=3$,
$K=6$



UNIT (Hyper-) SPHERE

- All positional vectors p_i and $\tilde{p}$ (corresponding to the points p_i and $\tilde{p}$) have UNIT LENGTH (are normalized) and have coordinates $0 \le x_1 \le 1$, $0 \le x_2 \le 1$, $0 \le x_3 \le 1$.
- (Since $N=3$,) "the (3) best" points p_i (vectors p_i) must be determined, to represent $\tilde{p}$ as a unique linear combination of the (3) best p_i's.

- INSTEAD OF USING THE METHOD OF HUGGINS & ZUCKER EXACTLY, DETERMINE (THREE) (LINEARLY INDEPENDENT) POINTS/VECTORS p_i THAT ARE **CLOSEST TO** $\tilde{p}$.

- **THUS:** COMPUTE THE SCALAR/DOT PRODUCTS BETWEEN ALL p_i AND $\tilde{p}$, i.e., compute $\langle p_i, \tilde{p} \rangle = p_i \bullet \tilde{p}$.
DETERMINE THE (3) POINTS THAT PRODUCE THE (3) LARGEST DOT PRODUCT VALUES. (Here: p_1, p_2, p_3)

$\Rightarrow$ "BEST REPRESENTION" OF $\tilde{p}$ IS: $\boxed{p = c_1 p_1 + c_2 p_2 + c_3 p_3}$ ■ BH

! → **USE:** c_i is close to 1 $\Leftrightarrow$ $\tilde{p}$ is type p_i.