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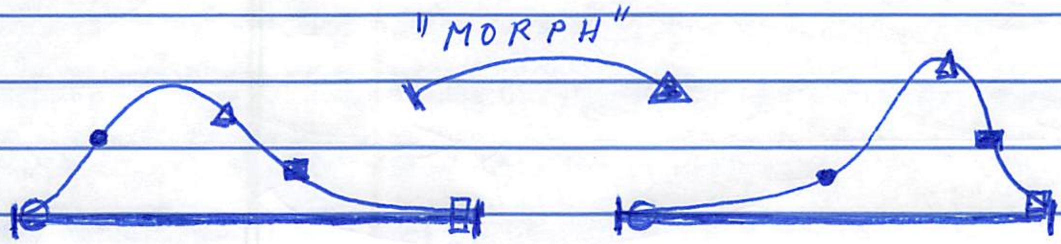
22 FEB 2016

- **PROBLEM:** "Good distance measure" for 2 histograms/ 2 histogram-based signatures

- **IDEA:** Answer this question:
  "How much 'energy' is needed to transform/morph one histogram into the other?" (→ Need for a meaningful 'energy' definition...)



histo 1   histo 2

⇒ Corresponding point pairs: ○, ●, △, ■, □
(→ How to define/establish a correspondence?)

- Resource: "Image Morphing, Thin-Plate Spline Model Computer Vision"
  http://grail.cs.washington.edu/projects/photomontage/

⇒ Define a thin-plate spline (TPS) deformation that "morphs" histo 1 into histo 2; the 'energy' associated with that deformation will then define a 'distance' between histo 1 and histo 2.

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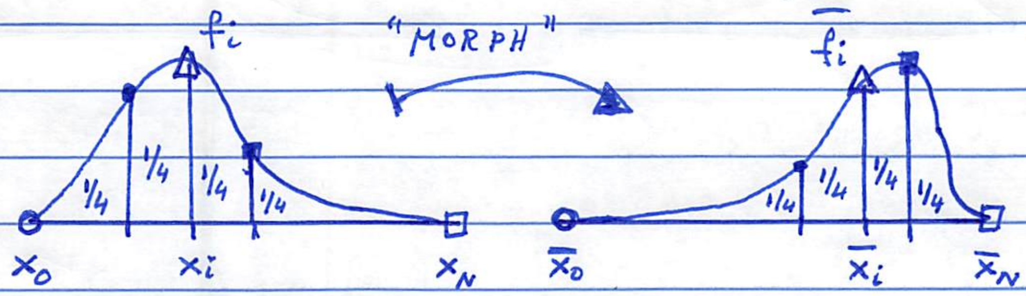
Defining a TPS-based Deformation

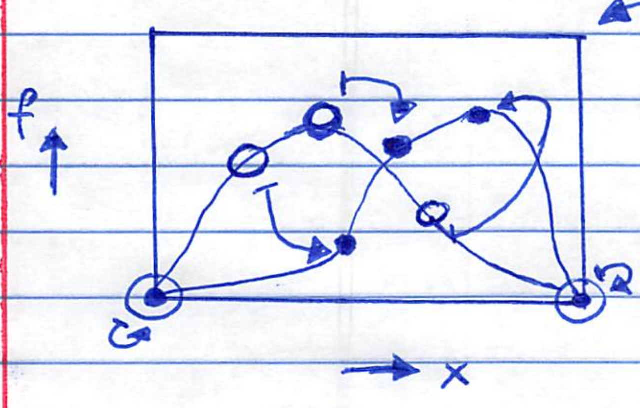
- Need to establish meaningful point pair correspondences automatically (="corresponding landmark points")

- Example 1: "Divide the area (= 1 for normalized histograms/distributions) under the 'histogram curve' into equal-size sub-areas." (→ The resulting split/break points on the x-axis define the needed corresponding point pairs.)



histo 1 histo 2

⇒ Corresponding point pairs: $\begin{pmatrix} x_i \\ f_i \end{pmatrix} \mapsto \begin{pmatrix} \bar{x}_i \\ \bar{f}_i \end{pmatrix}$, $i=0...N$



corresponding point pairs in "integrated (x, f)-view":

"o" histo 1 point

"•" histo 2 point

↦ mapping "o" to corresponding "•"

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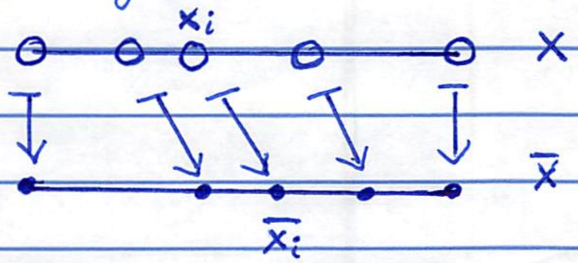
1) Only consider the "morphing" of **x-values to $\bar{x}$-values**:



$x_i \mapsto \bar{x}_i, \quad i = 0 \dots N$

(x_i-sequence and $\bar{x}_i$-sequence both monotonically increasing, in terms of values...)

$\Rightarrow$ Can determine TPS(*) (with basis function $T_i(x)$(**)) for mapping x_i-values to $\bar{x}_i$-values:

$$\sum_{i=0}^{N} c_i \, T_i(x_0) = \bar{x}_0$$

$$\dots$$

$$\sum_{i=0}^{N} c_i \, T_i(x_N) = \bar{x}_N$$

$\Rightarrow$ system of equations for unknown coefficients c_i:

$$\sum_{i=0}^{N} c_i \, T_i(x_j) = \bar{x}_j,$$

$$j = 0 \dots N$$

{(*) **NOTE-TPS**: Good references: (i) Fred Bookstein, "Principal Warps: Thin-Plate Splines and the Decomposition of Deformations"; (ii) Richard Franke, "Smooth Interpolation of Scattered Data by Local Thin Plate Splines"

(**) The basis function $T_i(x)$ is usually a radial basis function of the type $T_i(x) = d_i^2(x) \cdot \log(d_i(x))$, where
$$d_i^2(x) = (x - x_i)^2 .$$
}

2) Once the TPS-based "deformation of the x-axis to the $\bar{x}$-axis" is known, one can directly compute its associated '<u>energy</u>':

$$\underline{\text{Energy}(TPS)} = E(TPS) = E(\bar{x}) = \int_0^1 (x - \bar{x})^2 \, dx$$

(<u>Meaning</u>: The energy measures the sum of all 'displacements' from an x-value to its associated new $\bar{x}$-value.)

3) Consider the deformation as a combined x-value plus f-value deformation, requiring 2 deformation functions for x and f:

$$\left.\begin{array}{l} \sum_{i=0}^{N} \begin{pmatrix} c_i^x \\ c_i^f \end{pmatrix} \cdot T_i(x_0) = \begin{pmatrix} \bar{x}_0 \\ \bar{f}_0 \end{pmatrix} \\ \quad \cdots \\ \sum_{i=0}^{N} \begin{pmatrix} c_i^x \\ c_i^f \end{pmatrix} \cdot T_i(x_N) = \begin{pmatrix} \bar{x}_N \\ \bar{f}_N \end{pmatrix} \end{array}\right\} \Rightarrow \sum_{i=0}^{N} \begin{pmatrix} c_i^x \\ c_i^f \end{pmatrix} \cdot T_i(x_j) = \begin{pmatrix} \bar{x}_j \\ \bar{f}_j \end{pmatrix}, \quad j = 0 \ldots N$$

$\Rightarrow$ resulting energy:

$$E(\bar{x}, \bar{f}) = \int_0^1 \left\| \begin{pmatrix} x \\ f \end{pmatrix} - \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} \right\|^2 dx$$

/* measure of total displacement of */
/* (x, f)-tuples to new $(\bar{x}, \bar{f})$-tuples */ BH

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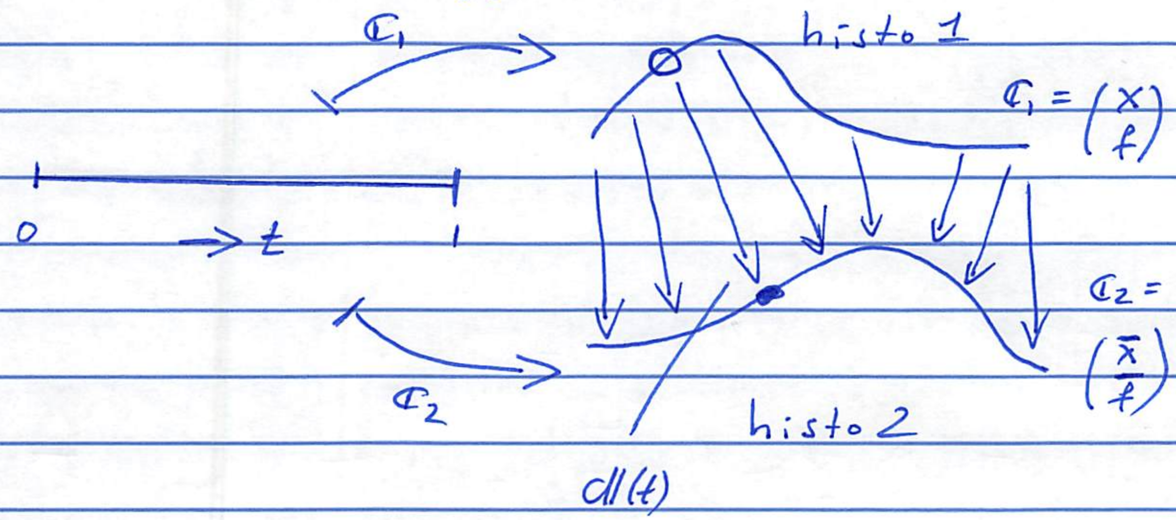
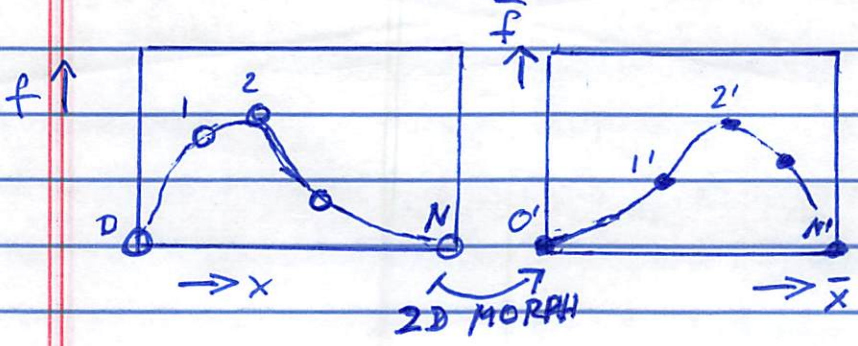
Other Possibilities to Measure Distance

- Consider histo 1 and histo 2 as 2 parametric curves, using the interval [0,1] as t-parameter domain, and using an arc length parametrization; then consider the sum total of displacement vectors (implied by corresponding point pairs) to define an energy-based distance:



$$\underline{\text{Distance}} = \int_0^1 \| d(t) \|^2 \, dt$$ "total displacement"

- Think of the deformation of histo 1 into histo 2 as a 2D mapping of the $\binom{x}{f}$-plane to the $\binom{\bar{x}}{\bar{f}}$-plane – and define an energy for a 2D-to-2D morph.



~ BH