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[• REF: Maxime DESCOTEAUX et al., "Bone Enhancement Filtering: Application..."]

Hessian and its eigenvalues – Local shape measure using least squares quadratic polynomial

→ Related topics:
- Quadric surfaces
- Sheets, tubes, blobs !
- Implicit surfaces

1) For a local neighborhood of voxels, compute the quadratic approximating polynomial (with f being "intensity"):

$$
\begin{aligned}
f(x,y,z) &= c_{200}x^2 + c_{020}y^2 + c_{002}z^2 \\
&\quad + c_{110}xy + c_{101}xz + c_{011}yz \\
&\quad + c_{100}x + c_{010}y + c_{001}z \qquad + c_{000} \\
&= \sum_{\substack{i,j,k \geq 0 \\ i+j+k=2}} c_{ijk}\, x^i y^j z^k
\end{aligned}
$$

!

2) The second derivatives of f define the Hessian H:

$$
\mathbb{H} = \begin{pmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{yz} & f_{zz} \end{pmatrix} = \begin{pmatrix} 2c_{200} & c_{110} & c_{101} \\ c_{110} & 2c_{020} & c_{011} \\ c_{101} & c_{011} & 2c_{002} \end{pmatrix}
$$

→ The matrix H is symmetric and has eigenvalues λ_1, λ_2 and λ_3 – which can have multiplicities larger than 1 and can be non-real.

→ The eigenvalues can be used to locally define an "implicit surface shape descriptor" – to characterize sheet, tube & blob shapes !

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■ The eigenvalues of the Hessian and "local shape"

→ H's eigenvalues are defined by

$$\det\begin{pmatrix} 2c_{200}-\lambda & c_{110} & c_{101} \\ c_{110} & 2c_{020}-\lambda & c_{011} \\ c_{101} & c_{011} & 2c_{002}-\lambda \end{pmatrix} = \underline{0}$$

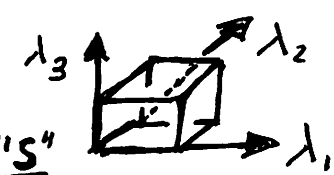
→ Considering only the case of real eigenvalues λ_i, $i=1...3$, one can order the λ_i based on value/absolute value.

! → IT SUFFICES TO CONSIDER THE EIGENVALUES TO CHARACTERIZE SHEET, TUBE & BLOB BEHAVIOR!

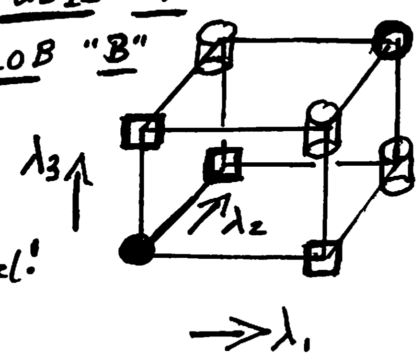
→ These are the "prototype cases" to be handled:

	$f(x,y,z) =$	λ_1	λ_2	λ_3
I	x^2	2	0	0
	y^2	0	2	0
	z^2	0	0	2
II	x^2+y^2	2	2	0
	x^2+z^2	2	0	2
	y^2+z^2	0	2	2
III	$x^2+y^2+z^2$	2	2	2
IV	0	0	0	0

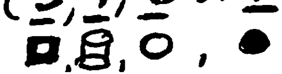
→ The EIGHT $(\lambda_1, \lambda_2, \lambda_3)$ tuples in this table can be viewed as the corners of a cube:



→ I) $x^2=c$, $y^2=c$, $z^2=c$ PLANES ≙ SHEETS "S"
II) $x^2+y^2=c$, $x^2+z^2=c$, $y^2+z^2=c$ CYLINDERS ≙ TUBES "T"
III) $x^2+y^2+z^2=c$ SPHERE ≙ BLOB "B"



→ The $(\lambda_1, \lambda_2, \lambda_3)$ eigenvalue space can be used to associate a local "shape classifier" (S, T, B or N [=noise]) with voxel!



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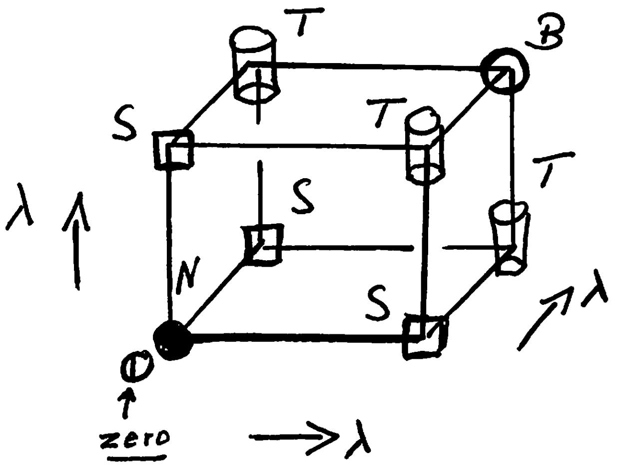
Local shape characterization via Hessian...

→ It is possible to distinguish between SHEET, TUBE, BLOB and NOISE as follows:



I) SHEET S:
1 λ-value "large", 2 λ-values "small"

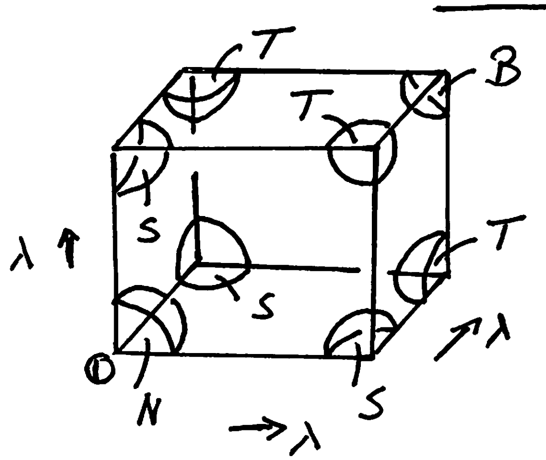
II) TUBE T:
2 λ-values "large", 1 λ-value "small"

III) BLOB B:
3 λ-values "large"

IV) NOISE N:
3 λ-values "very small"

⇓

- Can establish a "meaningful ε-neighborhood" around the S, T, B, N corners to define entire $(\lambda_1, \lambda_2, \lambda_3)$-regions in this continuous λ-parameter space to be viewed as S, T, B, N "regions":

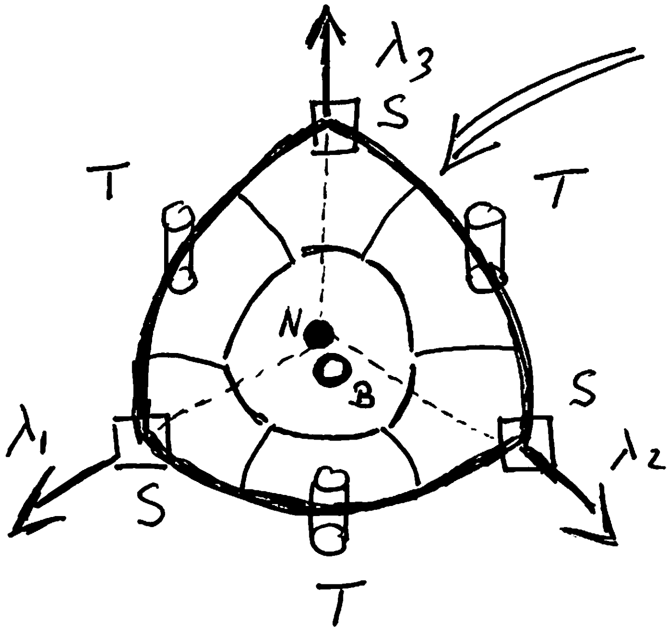


→ ISSUE: How to 'optimally' partition the $(\lambda_1, \lambda_2, \lambda_3)$-space into subregions that define S, T, B, N behavior 'well' ???

→ For example, parameter tuples $\lambda_1 \cong \lambda_2 \cong \lambda_3$ are on the line from 0 to B and ALL define a blob:
$\lambda_1 = \lambda_2 = \lambda_3 \Leftrightarrow$ spheres ■

Interpretation of Hessians eigenvalues...

→ Considering the case of only non-negative, real λ-values, one can associate regions on a spherical octant with different local shape behavior:



Octant subdivided into "tiles" associated with S, T, B behavior

→ A $(\lambda_1, \lambda_2, \lambda_3)$-tuple can be normalized, and it can be determined whether that tuple is in an S-, T- or B-tile!

[$\lambda = 0$ is special case]

Relationship to implicitly defined QUARIC SURFACES

Prototype	Surface
$x^2 = c$	Pair of Planes → SHEETS
$x^2 - y^2 = c$	Hyperbolic Cylinder
$x^2 + y^2 = c$	Cylinder (Circular) → TUBE
$x^2 - y^2 - z^2 = c$	Two-sheet Hyperboloid
$x^2 + y^2 - z^2 = c$	One-sheet Hyperboloid
$x^2 + y^2 + z^2 = c$	Sphere → BLOB

"Classification of Quadric Surfaces"

1. Identification of sheets, tubes and blobs is most important ⇒ eigenvalue analysis of Hessian can be reduced to only non-negative λ-values!
2. To identify the other quadric surface types, one must also consider negative and complex eigenvalues.

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