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2-25-2019

■ DENOISING – Considering Variation and Using Best Approximation ("BLaC")*

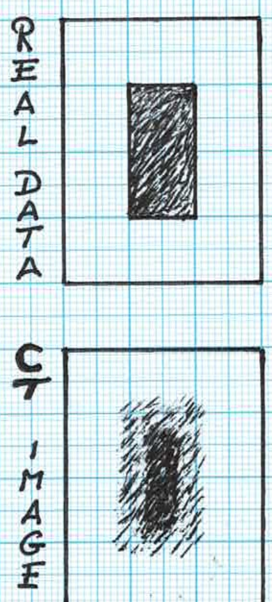
* BLaC = Blending of LINEAR and CONSTANT

→ Principle: "USE A MATHEMATICAL MODEL, A SPLINE CONSISTING OF CONSTANT AND LINEAR PIECES TO OPTIMALLY APPROXIMATE NOISY CT DATA OF OBJECTS WITH DISTINCT ('DISCONTINUOUS') DENSITIES."

→ Rationale:

- Our CT data results from imaging multiple objects with specific densities; each object can and should be modeled as a constant-density object (assuming that density is homogeneous in the interior of each object).
- Blurring and 'mixing' of density values occurs (typically) in the interface regions where 'sharp density discontinuities exist but are not preserved' (due to the imaging and reconstruction processes).

Example:



noisy; blurred boundary

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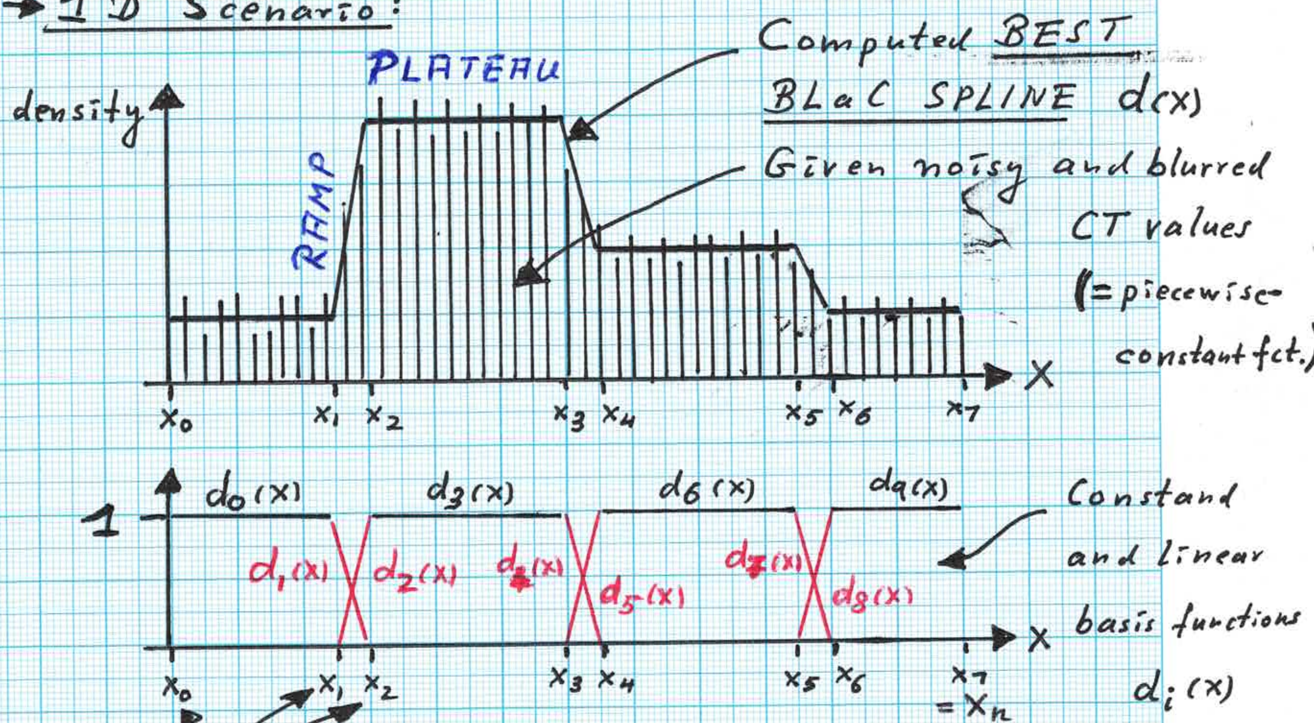
■ DENOISING — BLaC, Cont'd.

- *Each object* should be approximated with a *constant-value function*; *each interface* region (object-object interface or object-background interface) should be approximated with a *linear-ramp function*.

→ **1D Scenario:**



Knots / break points of BLaC spline (computed via variational measure associated with each given CT value):

$x_0, \ldots x_7 \Rightarrow d_0(x), \ldots d_9(x)$

⇒ *REGIONS:*

$[x_0, x_1]$ background

$[x_1, x_2]$ ramp

$[x_2, x_3]$ object 1

$[x_3, x_4]$ ramp

$[x_4, x_5]$ object 2

...

$[x_6, x_7]$ background

⇒ *USE BLaC basis fcts: LINEAR-CONST-LINEAR!* (Next pages...)

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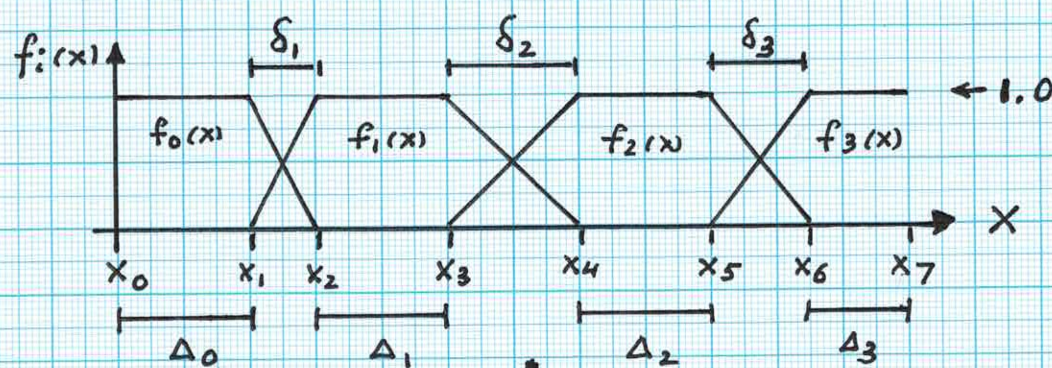
■ DENOISING – BLaC, Cont'd.

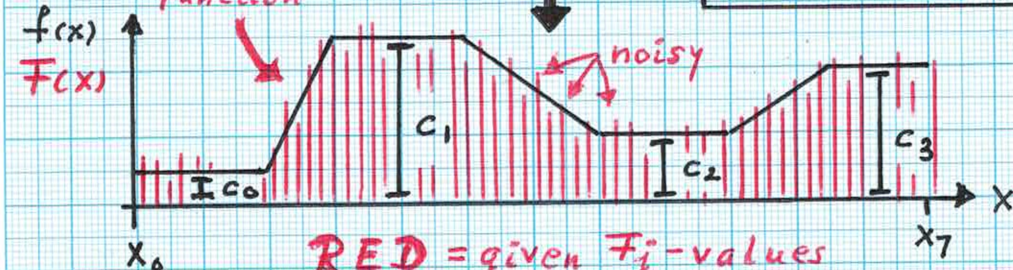
⇒ Establish notation for BLaC basis functions; define the needed inner products; and generate the resulting linear system:



$F(x)$: piecewise constant function

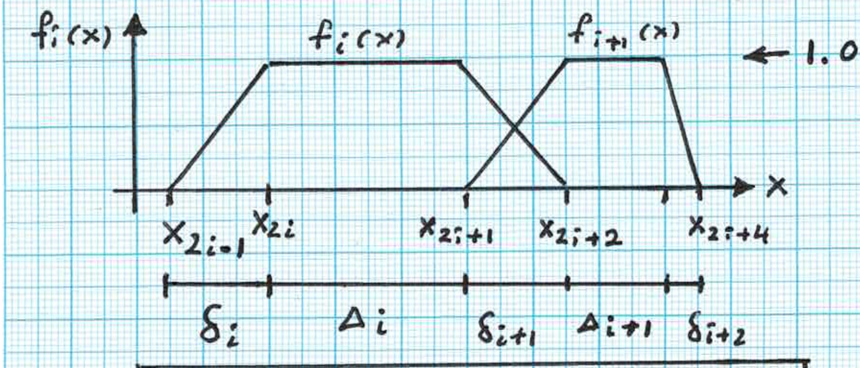
resulting approximation

$$\boxed{f(x) = \sum_{i=0}^{3} c_i f_i(x)}$$



= best possible approximation of given noisy density function with basis $\{f_i(x)\}$

⇒ Needed information – Inner products of basis functions; inner products of $F(x)$ and $f_i(x)$



$$\boxed{\langle F(x), f_i(x) \rangle = \int_{x_{2i-1}}^{x_{2i+2}} F(x) \cdot f_i(x)\, dx}$$

$$\langle f_i(x), f_i(x) \rangle = \frac{1}{3}\delta_i + \Delta_i + \frac{1}{3}\delta_{i+1}$$

$$\langle f_i(x), f_{i+1}(x) \rangle = \frac{1}{6}\delta_{i+1}$$

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■ DENOISING – BLaC, Cont'd.

⇒ Resulting linear system (for example on page 3):

$$\begin{bmatrix} \langle f_0, f_0\rangle & \langle f_0, f_1\rangle & 0 & 0 \\ \langle f_1, f_0\rangle & \langle f_1, f_1\rangle & \langle f_1, f_2\rangle & 0 \\ 0 & \langle f_2, f_1\rangle & \langle f_2, f_2\rangle & \langle f_2, f_3\rangle \\ 0 & 0 & \langle f_3, f_2\rangle & \langle f_3, f_3\rangle \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} \langle F, f_0\rangle \\ \langle F, f_1\rangle \\ \langle F, f_2\rangle \\ \langle F, f_3\rangle \end{bmatrix}$$

➡ Tridiagonal system!

$$\begin{bmatrix} 6\Delta_0 + 2\delta_1 & \delta_1 & 0 & 0 \\ \delta_1 & 2\delta_1 + 6\Delta_1 + 2\delta_2 & \delta_2 & 0 \\ 0 & \delta_2 & 2\delta_2 + 6\Delta_2 + 2\delta_3 & \delta_3 \\ 0 & 0 & \delta_3 & 2\delta_3 + 6\Delta_3 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = 6 \cdot \begin{bmatrix} \int_{x_0}^{x_2} F(x)\cdot f_0(x)\,dx \\ \int_{x_1}^{x_4} F(x)\cdot f_1(x)\,dx \\ \int_{x_3}^{x_6} F(x)\cdot f_2(x)\,dx \\ \int_{x_5}^{x_7} F(x)\cdot f_3(x)\,dx \end{bmatrix}$$

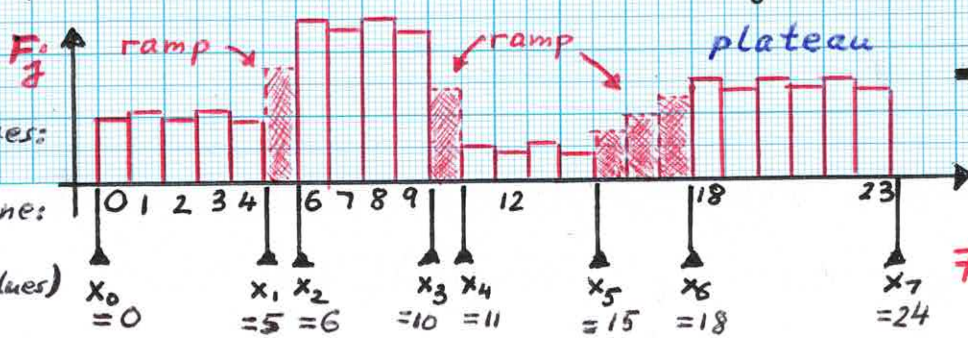
• **NOTE:** THIS SYSTEM BECOMES THE SYSTEM FOR A CONSTANT (LINEAR) SPLINE WHEN ALL $\delta_i = 0$ ($\Delta_i = 0$).

⇒ Must establish relationship to given discrete $\{F_j\}$ data:

• Given: F_j, $j = 0, \ldots, (N-1)$

• Implied x-values: $x = 0, \ldots, N$

• 'Knots' of spline: $x_0, \ldots, x_7$ (having INT values)



➡ $\int F(x) \cdot f_i(x)\,dx$

$F(x)$: piecewise-constant function!

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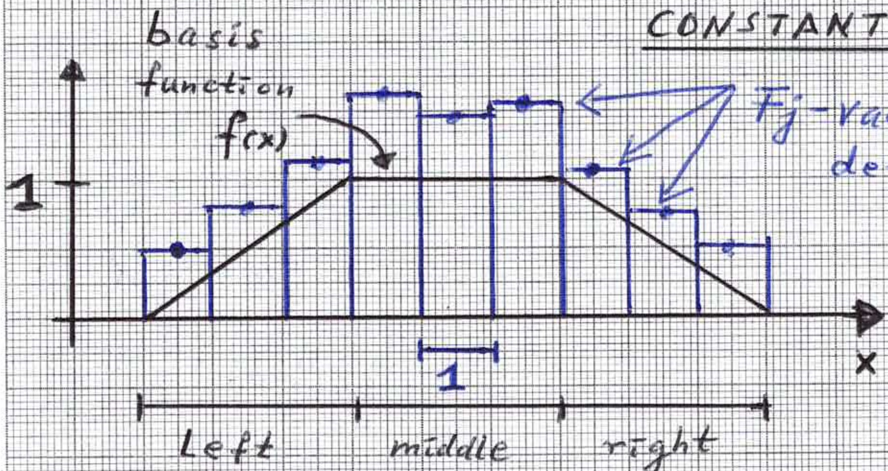
DENOISING – BLaC, Cont'd.

⇒ Needed Information – Inner products of basis functions and PIECEWISE-CONSTANT function F:



$$\langle F, f\rangle = \langle F, f\rangle^{left} + \langle F, f\rangle^{middle} + \langle F, f\rangle^{right}$$

↓ L — F_j-values, indexed $0, \dots, L-1$

↓ M — F_j-values, indexed $0, \dots, M-1$

↓ R — F_j-values, indexed $0, \dots, R-1$

$$\langle F, f\rangle^{left} = \frac{1}{2L} \sum_{j=0}^{L-1} (2j+1) F_j = \int_{Left} F \cdot f \, dx$$

$$\langle F, f\rangle^{middle} = \sum_{j=0}^{M-1} F_j = \int_{middle} F \cdot f \, dx$$

$$\langle F, f\rangle^{right} = \sum_{j=0}^{R-1} F_j - \frac{1}{2R} \sum_{j=0}^{R-1} (2j+1) F_j$$

$$= \frac{1}{2R} \sum_{j=0}^{R-1} (2(R-j)-1) F_j = \int_{right} F \cdot f \, dx$$

➡ ALL inner products $\langle F(x), f_i(x)\rangle$ can be computed exactly via these formulas.

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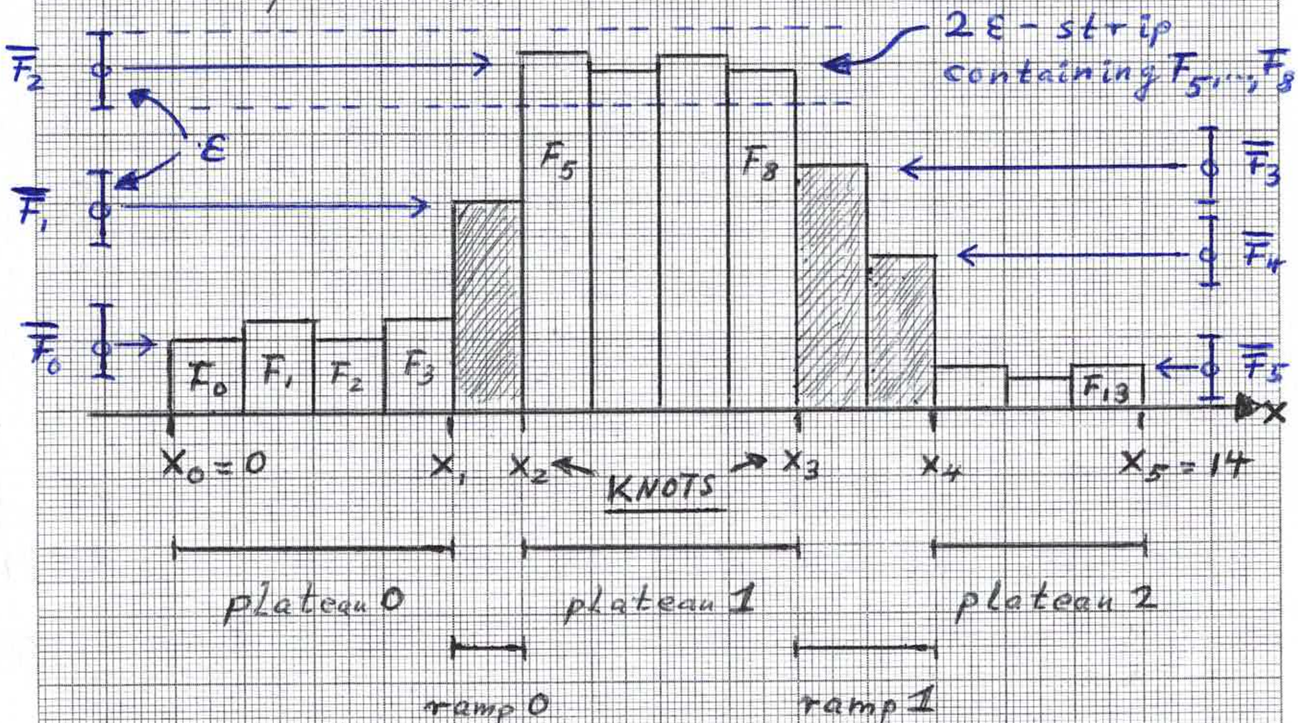
DENOISING – BLaC, Cont'd.

⇛ Determining **KNOTS** of BLaC spline – a greedy, **brute-force** method (not generally producing optimal knots):

• Example:



$\overline{F}_0 = \frac{1}{4}(F_0 + F_1 + F_2 + F_3)$ = mean of values $F_0, \dots, F_3$, such that $|F_j - \overline{F}_0| < \varepsilon$, $j = 0, \dots, 3$

$\overline{F}_1 = \frac{1}{1} F_4 = \dots$

$\overline{F}_5 = \frac{1}{3}(F_{11} + F_{12} + F_{13}) = \dots$

• **A**ssumption: Given data "start and end with plateaus."
(It is possible to force the BLaC spline to start/end with a constant spline segment – or to add values before the first and after the last given F-values implying constant segments.)

DENOISING – BLaC, Cont'd.

⇒ Determining KNOTS of BLaC spline – by defining subsets of F_j-values belonging to plateaus and ramps

- **INPUT:** $F_0, F_1, F_2, \ldots$; tolerance ε
- **OUTPUT:** knots $x_0 = 0, x_1, x_2, \ldots$
- **ALGORITHM:** DEFINE SUBSETS OF GIVEN F_j-VALUES BY FINDING SUCCESSIVE F_j-VALUES OF PLATEAUS AND RAMPS.

→ Process the given F_j-values from left to right.

→ In each step, determine the maximal number K of F_j-values with a mean such that

$$\left| F_j - \frac{1}{K} \sum F_j \right| < \varepsilon .$$

→ For each of the resulting subsets, decide whether it consists of F_j-values defining a plateau or (part of) a ramp. [NOTE: A ramp can be defined by consecutive subsets with each subset containing one F_j-value.]

→ Place KNOTS at the ends of the computed plateau and ramp subsets of F_j-values; use the knots $x_0, x_1, x_2, \ldots$ for the computation of the BLaC spline.

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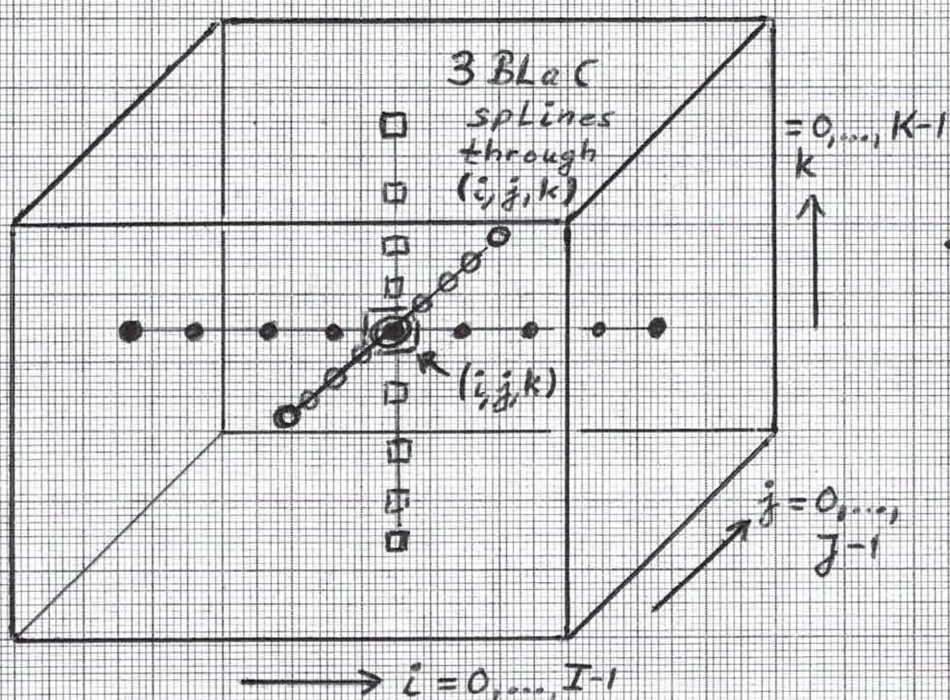
■ DENOISING – BLaC, Cont'd.

⇒ Computing UNIVARIATE BLaC splines for volume data:



- Given:
  (I·J·K) voxel data
- Compute:
  i) J·K univariate BLaC splines in i-direction
  ii) I·K univariate BLaC splines in j-direction
  iii) I·J univariate BLaC splines in k-direction

⇒ DENOISING: REPLACE VALUE OF VOXEL (i,j,k) BY AVERAGE VALUE OF THE THREE UNIVARIATE BLaC SPLINES EVALUATED AT THE CENTER OF VOXEL (i,j,k). ●

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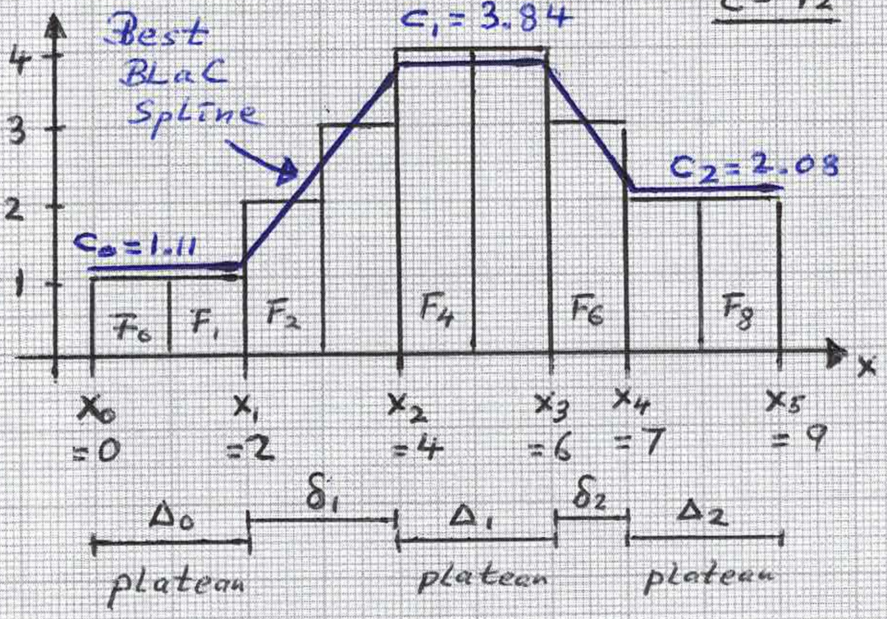
DENOISING - BLaC, Cont'd.

⇒ Simple example:

$\varepsilon = 1/2$



• Given:

$F_0 = F_1 = 1$

$F_2 = 2$

$F_3 = 3$

$F_4 = F_5 = 4$

$F_6 = 3$

$F_7 = F_8 = 2$

➜ piecewise constant function $F(x)$

• Resulting linear system for BLaC spline:

$$\begin{bmatrix} 6\Delta_0 + 2\delta_1 & \delta_1 & 0 \\ \delta_1 & 2\delta_1 + 6\Delta_1 + 2\delta_2 & \delta_2 \\ 0 & \delta_2 & 2\delta_2 + 6\Delta_2 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \langle F, f_0 \rangle \\ \langle F, f_1 \rangle \\ \langle F, f_2 \rangle \end{bmatrix} \cdot 6$$

• Here: $\Delta_0 = 2 = \Delta_1 = \Delta_2$

$\delta_1 = 2, \quad \delta_2 = 1$

$\langle F, f_0 \rangle = 17/4$

$\langle F, f_1 \rangle = 49/4$

$\langle F, f_2 \rangle = 11/2$

$$\Rightarrow \begin{bmatrix} 16 & 2 & 0 \\ 2 & 18 & 1 \\ 0 & 1 & 14 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = 6 \begin{bmatrix} 17/4 \\ 49/4 \\ 11/2 \end{bmatrix}$$

$\Rightarrow \underline{\underline{c_0 = 1.11, \; c_1 = 3.84, \; c_2 = 2.08}}$

~ BH