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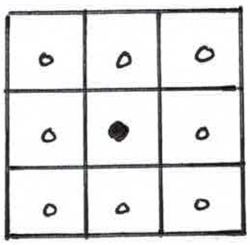
7/8/2019

■ LDAC – Using a Local Polynomial Approximation to Estimate Directional Differential Behavior



3-by-3 data stencil for bi-quadratic

→ Compute BI-QUADRATIC polynomial:

$$I(x,y) = c_{00} + c_{10}x + c_{01}y + c_{11}xy + c_{20}x^2 + c_{02}y^2 + c_{21}x^2y + c_{12}xy^2 + c_{22}x^2y^2$$

$$= \sum_{j=0}^{2}\sum_{i=0}^{2} c_{i,j}\,x^i y^j$$

/* obtained by */
/* interpolating */
/* 9 I-values */

→ Gradient of $I(x,y)$:

$$\nabla I = \left(c_{10} + c_{11}y + 2c_{20}x + 2c_{21}xy + c_{12}y^2 + 2c_{22}xy^2,\; c_{01} + c_{11}x + 2c_{02}y + 2c_{12}xy + c_{21}x^2 + 2c_{22}x^2y \right)$$
$$= (g_x, g_y)$$

→ Can compute ANY NUMBER of directional derivatives (anywhere in this local neighborhood):

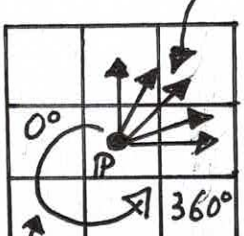
$D_{\vec{dl}}(\mathbb{P}) =$ [redacted] $\nabla I(\mathbb{P}) \cdot \vec{dl}$

→ $\vec{dl}$ normalized direction vector $\begin{pmatrix} d_x \\ d_y \end{pmatrix}$

→ $\mathbb{P}$ any point $\begin{pmatrix} x \\ y \end{pmatrix}$

→ $\nabla I(\mathbb{P}) = (g_x(\mathbb{P}), g_y(\mathbb{P}))$

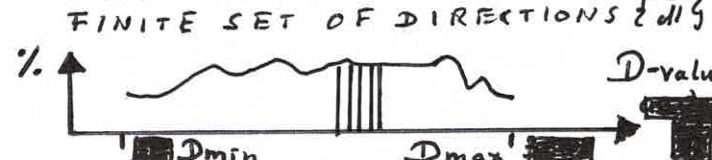
finite set of direction vectors $\vec{dl}$



direction encoded by angle 0°...360°

(• NOTE: $D_{-\vec{dl}}(\mathbb{P}) = -D_{\vec{dl}}(\mathbb{P})$)

→ GENERATE FOR A SET OF POINTS/PIXELS $\{\mathbb{P}\}$ A HISTOGRAM THAT DEFINES THE DISTRIBUTION OF DERIVATIVE VALUES $\{D_{\vec{dl}}\}$ FOR A FINITE SET OF DIRECTIONS $\{\vec{dl}\}$



■ LDAC - cont'd.

➜ Second directional derivative and Hessian:

$$I_{xx} = 2c_{20} + 2c_{21}y + 2c_{22}y^2$$

$$I_{yy} = 2c_{02} + 2c_{12}x + 2c_{22}x^2$$

$$I_{xy} = I_{yx} = c_{11} + 2c_{21}x + 2c_{12}y + 4c_{22}xy$$

$$\Rightarrow \text{Hessian} \quad H = \begin{pmatrix} I_{xx} & I_{xy} \\ I_{yx} & I_{yy} \end{pmatrix}$$

$\Rightarrow$ Second directional derivative:

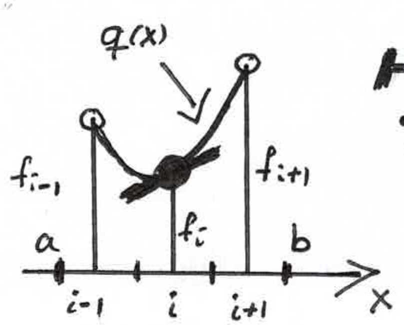
$$D^2_{\vec{dl}}(\mathbb{P}) = \vec{dl}^{\,T} \cdot H \cdot \vec{dl} = (d_x, d_y) \cdot H \cdot \begin{pmatrix} d_x \\ d_y \end{pmatrix}$$

➜ GENERATE HISTOGRAM OF SECOND DERIVATIVE VALUES...

• PRINCIPLE / USE:

The set of pixels constituting an object/segment of a specific material define histograms capturing distributions of intensity values and first and second directional derivative values of intensity.

Therefore: STORED LDAC HISTOGRAMS OF "DANGEROUS" MATERIALS CAN BE COMPARED WITH HISTOGRAMS OF MATERIALS FOUND IN A NEW IMAGE! ~



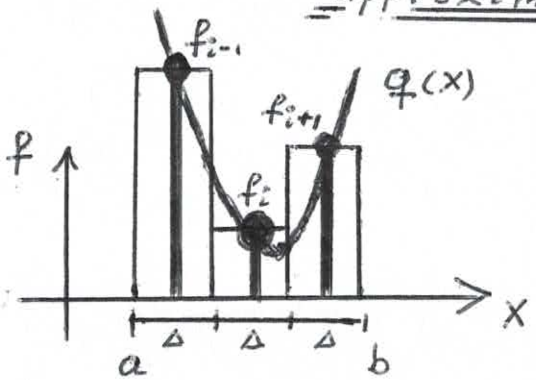
• ISSUE: Knowing the analytical definition of an interpolant $q(x)$, one can describe the local derivative behavior by:

(1) $D = q'(i)$ ← point

or (2) $D = \int_a^b |q'(x)|\, dx$ ← integral

> measure

LDAC – Local least-squares polynomial approximations for INTEGRAL properties



→ q interpolates values f_{i-1}, f_i and f_{i+1} ;

$$q(x) = c_0 + c_1 x + c_2 x^2$$

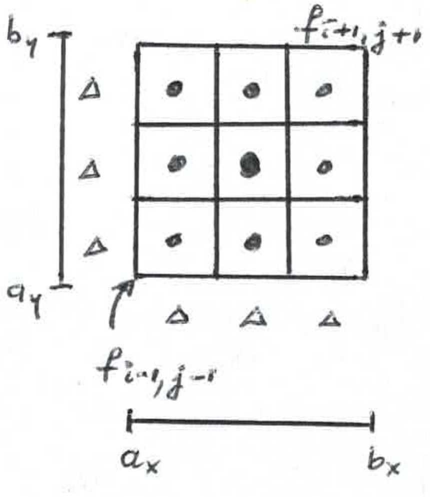
⇒ Compute integral properties of $q(x)$:

$$\text{(i)} \quad \overline{q} = \frac{1}{3\Delta} \int_{x=a}^{b} q(x)\,dx$$

$$\text{(ii)} \quad \overline{q'} = \frac{1}{3\Delta} \int_{x=a}^{b} q'(x)\,dx$$

$$\text{(iii)} \quad \overline{q''} = \frac{1}{3\Delta} \int_{x=a}^{b} q''(x)\,dx$$

} 3 "integral averages of differential behavior"



→ bivariate / 2D case:

q is a least squares quadratic approximation of 9 values $f_{i-1,j-1}, \ldots, f_{i+1,j+1}$;

$$q(x,y) = c_{00} + c_{10}x + c_{01}y + c_{11}xy + c_{20}x^2 + c_{02}y^2 ,$$

$$\Rightarrow \nabla q(x,y) = (c_{10} + c_{11}y + 2c_{20}x,\; c_{01} + c_{11}x + c_{02}2y), \quad H = \begin{pmatrix} 2c_{20} & c_{11} \\ c_{11} & 2c_{02} \end{pmatrix}$$

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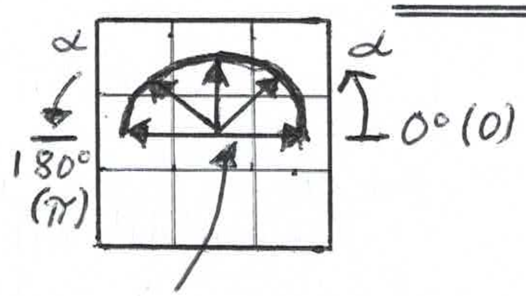
LDAC – INTEGRAL properties – cont'd.

⇒ Compute integral properties of $q(x,y)$:

(i) $$\overline{q} = \frac{1}{q\Delta^2} \int_{y=a_y}^{b_y} \int_{x=a_x}^{b_x} q(x,y)\, dx\, dy$$



(ii) first derivative property:

compute DIRECTIONAL first derivative average:

direction $dl = dl(\alpha) = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} = \begin{pmatrix} c\alpha \\ s\alpha \end{pmatrix}$,

$\alpha = 0 \ldots \pi$

$\nabla q(x,y) = (q_x, q_y)$

directional derivative $D_{dl}(x,y) = q_x c\alpha + q_y s\alpha$;

(ii) $$\overline{q'} = \frac{1}{q\Delta^2\pi} \int_{\alpha=0}^{\pi} \int_{y=a_y}^{b_y} \int_{x=a_x}^{a_y} q_x(x,y)\, c\alpha + q_y(x,y)\, s\alpha \; dx\, dy\, d\alpha$$

(iii) second derivative property:

Compute DIRECTIONAL second derivative average:

$$D^2_{dl}(x,y) = dl^T H\, dl = (c\alpha, s\alpha) \begin{pmatrix} 2c_{20} & c_{11} \\ c_{11} & 2c_{02} \end{pmatrix} \begin{pmatrix} c\alpha \\ s\alpha \end{pmatrix}$$

$$= \ldots = 2\,(c_{20}\, c^2\alpha + c_{11}\, c\alpha\, s\alpha + c_{02}\, s^2\alpha) ;$$

$$\overline{q''} = \frac{1}{q\pi\Delta^2} \int_{\alpha=0}^{\pi} \int_{y=a_y}^{b_y} \int_{x=a_x}^{b_x} 2\,(c_{20}\, c^2\alpha + c_{11}\, c\alpha\, s\alpha + c_{02}\, s^2\alpha)\, dx\, dy\, d\alpha$$

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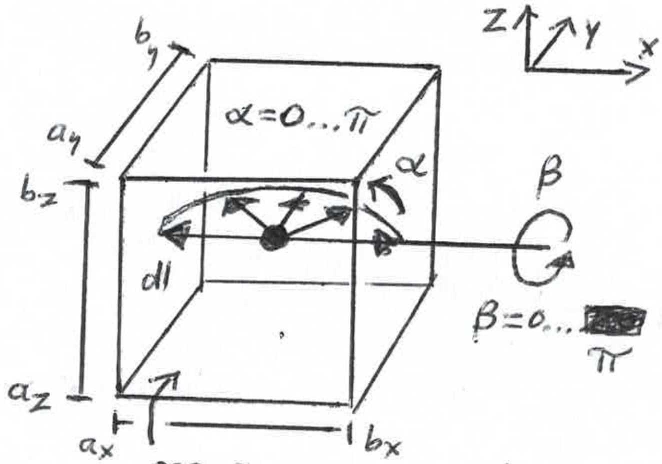
■ LDAC – INTEGRAL properties – cont'd.



- region occupied by 3-by-3-by-3 = 27 voxels
- values of α, β cover semi-sphere (defining dl)

→ Volumetric / 3D case:

Least squares quadratic approximation given by

$$\underline{q(x,y,z)} = c_{000} + c_{100}x + c_{010}y + c_{001}z + \ldots + c_{110}xy + \ldots + c_{002}z^2$$

$$= \sum_{\substack{i+j+k \le 2 \\ i,j,k \ge 0}} c_{i,j,k}\, x^i y^j z^k$$

$$\Rightarrow \nabla q(x,y,z) = \big(c_{100} + c_{110}y + c_{101}z + 2c_{200}x,\; c_{010} + c_{110}x + c_{011}z + 2c_{020}y,\; c_{001} + c_{101}x + c_{011}y + 2c_{002}z \big),$$

$$H = \begin{pmatrix} 2c_{200} & c_{110} & c_{101} \\ c_{110} & 2c_{020} & c_{011} \\ c_{101} & c_{011} & 2c_{002} \end{pmatrix}$$

$$(i) \quad \overline{q} = \frac{1}{27\Delta^3} \int_{z=a_z}^{b_z} \int_{y=a_y}^{b_y} \int_{x=a_x}^{b_x} q(x,y,z)\, dx\, dy\, dz$$

{ NOTE: must rotate $\begin{pmatrix} c\alpha \\ s\alpha \\ 0 \end{pmatrix}$ around x-axis by β:

$$dl = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c\beta & -s\beta \\ 0 & s\beta & c\beta \end{pmatrix} \begin{pmatrix} c\alpha \\ s\alpha \\ 0 \end{pmatrix} = \begin{pmatrix} c\alpha \\ s\alpha\, c\beta \\ c\alpha\, s\beta \end{pmatrix}, \quad \alpha = 0 \ldots \pi,\; \beta = 0 \ldots \pi \;\}$$

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■ **LDAC – cont'd.**

$\Rightarrow$ gradient $\nabla q = (q_x, q_y, q_z)$

direction $dl = \begin{pmatrix} c\alpha \\ s\alpha\, c\beta \\ c\alpha\, s\beta \end{pmatrix}$

directional derivative $D_{dl} = q_x\, c\alpha + q_y\, s\alpha c\beta + q_z\, c\alpha s\beta$

(ii) $$\overline{q'} = \frac{1}{27\pi^2\Delta^3} \int_{\beta=0}^{\pi} \int_{\alpha=0}^{\pi} \int_{z=a_z}^{b_z} \int_{y=a_y}^{b_y} \int_{x=a_x}^{b_x} D_{dl}(x,y,z,\alpha,\beta)\, dx\, dy\, dz\, d\alpha\, d\beta$$

– Second derivative property:

Compute DIRECTIONAL second derivative average:

$$\underline{D^2_{dl}(x,y,z)} = dl^T H\, dl = (c\alpha, s\alpha c\beta, c\alpha s\beta) \begin{pmatrix} 2c_{200} & c_{110} & c_{101} \\ c_{110} & 2c_{020} & c_{011} \\ c_{101} & c_{011} & 2c_{002} \end{pmatrix} \begin{pmatrix} c\alpha \\ s\alpha c\beta \\ c\alpha s\beta \end{pmatrix}$$

$$= \ldots = \underline{D^2(x,y,z,\alpha,\beta)} \;;$$

$$\overline{q''} = \frac{1}{27\pi^2\Delta^3} \int_{\beta=0}^{\pi} \int_{\alpha=0}^{\pi} \int_{z=a_z}^{b_z} \int_{y=a_y}^{b_y} \int_{x=a_x}^{b_x} D^2(x,y,z,\alpha,\beta)\, dx\, dy\, dz\, d\alpha\, d\beta$$

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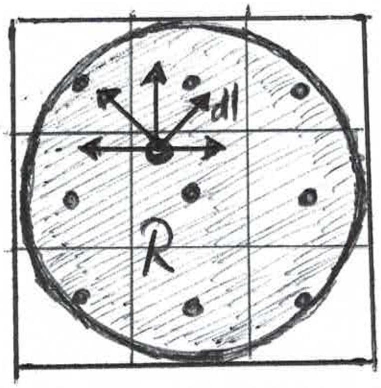
LDAC - cont'd.

SUMMARY:
- Compute local quadratic approximation
- Consider function value and directional first and second derivatives
- Compute averages of these differential properties by integrating over a region



(R has area $|R|$.)

→ $\{\bullet\}$ given values used to compute least squares approximation $q(x,y) = \sum_{\substack{i+j \le 2 \\ i,j \ge 0}} c_{ij} x^i y^j$

→ R = region over which properties of $q(x,y)$ are considered

→ $\{dl\}$ set of direction vectors (unit length) for directional derivatives, $dl = dl(\alpha) = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} = \begin{pmatrix} c\alpha \\ s\alpha \end{pmatrix}$, $\alpha \in [0, \pi]$

⇒ INTEGRAL LDAC PROPERTIES:

(i) $\overline{q} = \frac{1}{|R|} \int_R q(x,y)\, dx\, dy$

(ii) $\overline{q'} = \frac{1}{\pi |R|} \int_{\alpha=0}^{\pi} \int_R |D_{dl}|\, dx\, dy\, d\alpha$, where

$D_{dl} = (q_x(x,y), q_y(x,y)) \begin{pmatrix} c\alpha \\ s\alpha \end{pmatrix}$, $\nabla q = (q_x, q_y)$

(iii) $\overline{q''} = \frac{1}{\pi |R|} \int_{\alpha=0}^{\pi} \int_R |D^2_{dl}|\, dx\, dy\, d\alpha$, where

$D^2_{dl} = (c\alpha, s\alpha\ \text{[redacted]}) \begin{pmatrix} q_{xx} & q_{xy} \\ q_{yx} & q_{yy} \end{pmatrix} \begin{pmatrix} c\alpha \\ s\alpha \end{pmatrix}$ $\tilde{\approx}_{BH}$ ⇒ GENERALIZE TO 3D CASE.

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LDAC – cont'd.

SUMMARY FOR 3D CASE:

$$q(x,y,z) = \sum_{\substack{i+j+k \leq 2 \\ i,j,k \geq 0}} c_{ijk}\, x^i y^j z^k \qquad \text{approximation}$$

$$\nabla q = (q_x, q_y, q_z) \qquad \text{gradient}$$

$$H = \begin{pmatrix} q_{xx} & q_{xy} & q_{xz} \\ q_{yx} & q_{yy} & q_{yz} \\ q_{zx} & q_{zy} & q_{zz} \end{pmatrix} \qquad \text{Hessian}$$

$$dl = dl(\alpha,\beta) = \begin{pmatrix} c\alpha \\ s\alpha\ c\beta \\ s\alpha\ s\beta \end{pmatrix}, \quad \alpha,\beta \in [0,\pi] \qquad \text{normalized direction vector}$$

R = volumetric region in 3D space bounded by sphere

⇒ INTEGRAL PROPERTIES:

(i) $\overline{q} = \frac{1}{|R|} \int_R q(x,y,z)\, dx\, dy\, dz$

(ii) $\overline{q'} = \frac{1}{\pi^2 |R|} \int_{\beta=0}^{\pi} \int_{\alpha=0}^{\pi} \int_R |D_{dl}|\, dx\, dy\, dz\, d\alpha\, d\beta$,

where $D_{dl} = \ldots$

(iii) $\overline{q''} = \frac{1}{\pi^2 |R|} \int_{\beta=0}^{\pi} \int_{\alpha=0}^{\pi} \int_R |D^2_{dl}|\, dx\, dy\, dz\, d\alpha\, d\beta$,

where $D^2_{dl} = \ldots$

~ BH

⇒ USE GAUSSIAN QUDRATURE FOR EFFICIENT COMPUTATION!