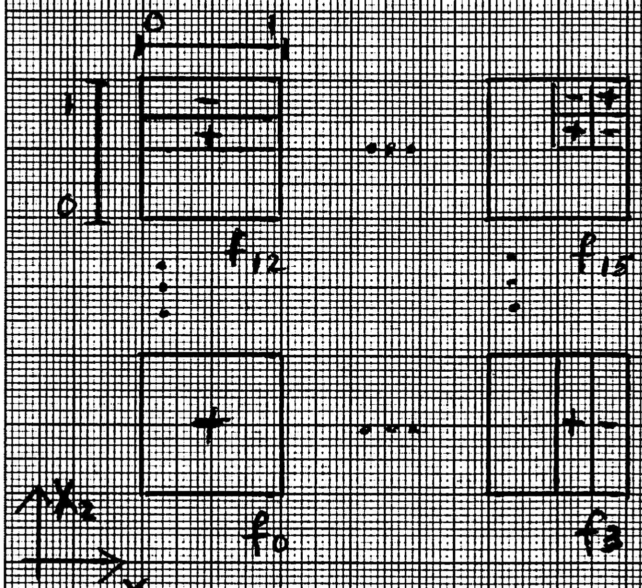
Bernd Hamann — 1 — 8/16/2019

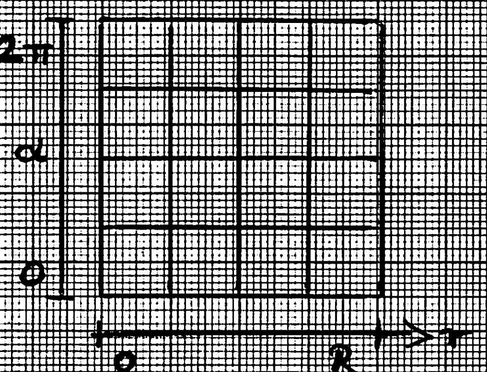
Stratovan

HAAR WAVELETS & BEST APPROXIMATION IN AN ORIENTATION-INVARIANT WAY

1) Example: Consider a bivariate tensor product (TP) Haar wavelet basis of 16 basis functions

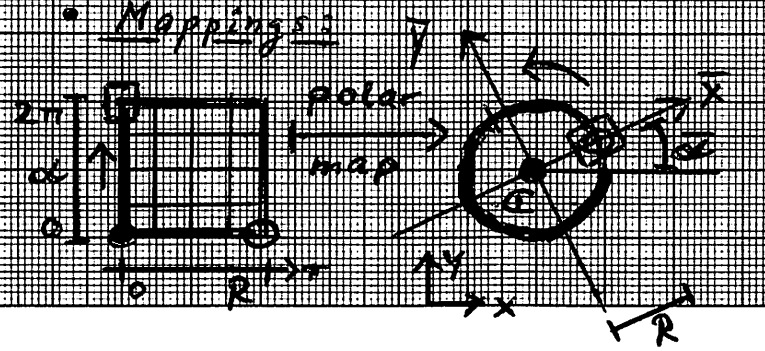


⇒ The 16 basis functions $f_0, ..., f_{15}$ are mapped to (r, α)-space:



⇒ 4x4 grids

- The (r, α)-space defines the parameter space of a DISK (with radius R) in (x, y)-space.
- The 16 basis functions are defined in the "rectangular" (r, α)-space, defining an ORTHONORMAL basis.
- Mappings:



→ pixels oriented according to (x,y) system

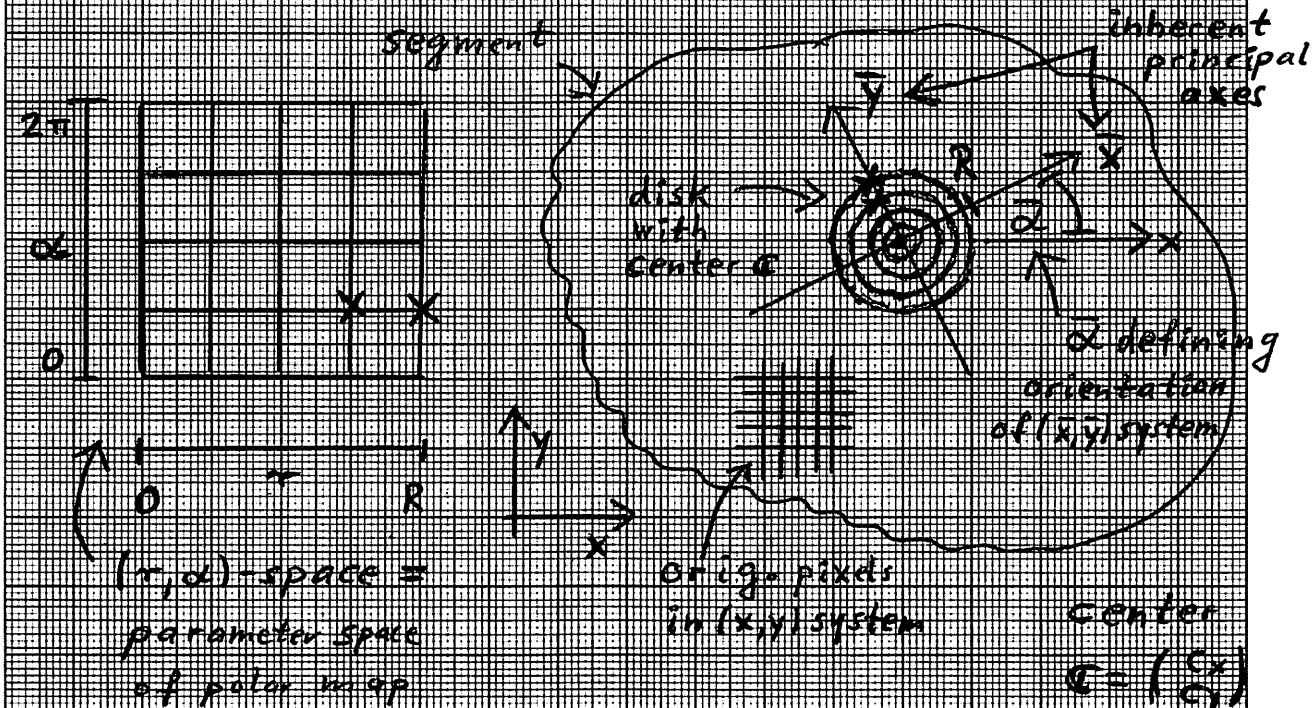
→ disk has center C and radius R.

→ $(\bar{x}, \bar{y})$ system defined by pixel-value induced ORIENTATION given by $\bar{\alpha}$

Stratovan

• HAAR WAVELETS – Cont'd.

2) Polar (or Log-polar) Coordinate Transform.



POLAR MAP

$$\boxed{\begin{pmatrix} r \\ \alpha \end{pmatrix} \longmapsto \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_x + r\cos(\alpha+\bar{\alpha}) \\ c_y + r\sin(\alpha+\bar{\alpha}) \end{pmatrix}} \quad \begin{matrix} 0 \le r \le R \\ 0 \le \alpha \le 2\pi \end{matrix}$$

$$\Rightarrow \text{Jacobian} = \underline{\underline{J}} = \det\begin{pmatrix} x_r & x_\alpha \\ y_r & y_\alpha \end{pmatrix}$$

$$= \begin{vmatrix} \cos(\alpha+\bar{\alpha}) & -r\sin(\alpha+\bar{\alpha}) \\ \sin(\alpha+\bar{\alpha}) & r\cos(\alpha+\bar{\alpha}) \end{vmatrix}$$

$$= r\left(\sin^2(\alpha+\bar{\alpha}) + \cos^2(\alpha+\bar{\alpha})\right)$$

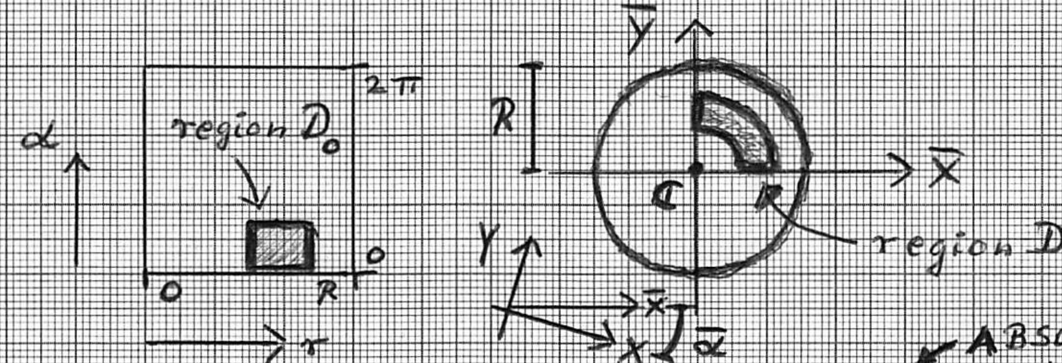
$$\underline{\underline{= r}}$$

• NOTE: J is needed for computing BEST APPROXIMATION of pixel values inside disk with Haar wavelet basis functions.

HAAR WAVELETS – Cont'd.

3) Change-of-variables Theorem

[→ Computing integrals (inner products)]



$r = |J|$ ← ABSOLUTE VALUE OF JACOBIAN DET.

$$\boxed{\int_D f(x,y)\,dx\,dy = \int_{D_0} r \cdot f(x(r,\alpha), y(r,\alpha))\,dr\,d\alpha}$$

$D_0 = [r_{min}, r_{max}] \times [\alpha_{min}, \alpha_{max}]$

4) Best Approximation

- Goal: Compute best approximation, in the least squares sense, for the GIVEN FUNCTION (= piecewise constant function defined by given pixel values inside disk) $F(x,y)$ in HAAR WAVELET BASIS considering the DISK DOMAIN!

⇒ BEST APPROX $A(x,y) = \sum_{i=0}^{15} c_i f_i(x,y)$

⇒ c_i-values capture multi-scale behavior!

Bernd Hamann — 4 — 8/19/2019
Stratovan

HAAR WAVELETS – Cont'd.

4) Best Approximation

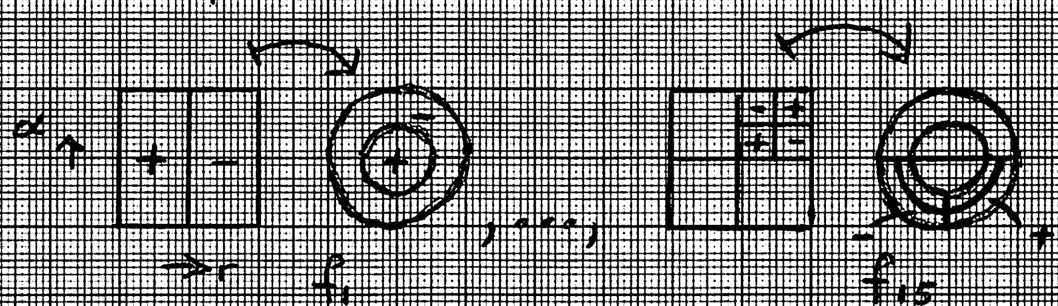
Solve normal equations to obtain c_i-values:

$$\begin{pmatrix} \langle f_0, f_0 \rangle & \cdots & \langle f_0, f_{15} \rangle \\ \vdots & & \vdots \\ \langle f_{15}, f_0 \rangle & \cdots & \langle f_{15}, f_{15} \rangle \end{pmatrix} \begin{pmatrix} c_0 \\ \vdots \\ c_{15} \end{pmatrix} = \begin{pmatrix} \langle F, f_0 \rangle \\ \vdots \\ \langle F, f_{15} \rangle \end{pmatrix}$$

→ $\langle\, , \rangle$ = inner product of 2 functions over DISK

→ To compute inner products over disk domain in (x,y)-space, the Haar wavelet basis functions, originally represented as tensor product functions in (r,α)-space, must be mapped to "deformed domains" in the disk.

Example:



→ The "standard" tensor product Haar wavelet basis is an ORTONORMAL BASIS: $\langle f_i, f_j \rangle = \delta_{i,j}$

Here: The disk domain makes necessary to "adjust" basis functions to achieve

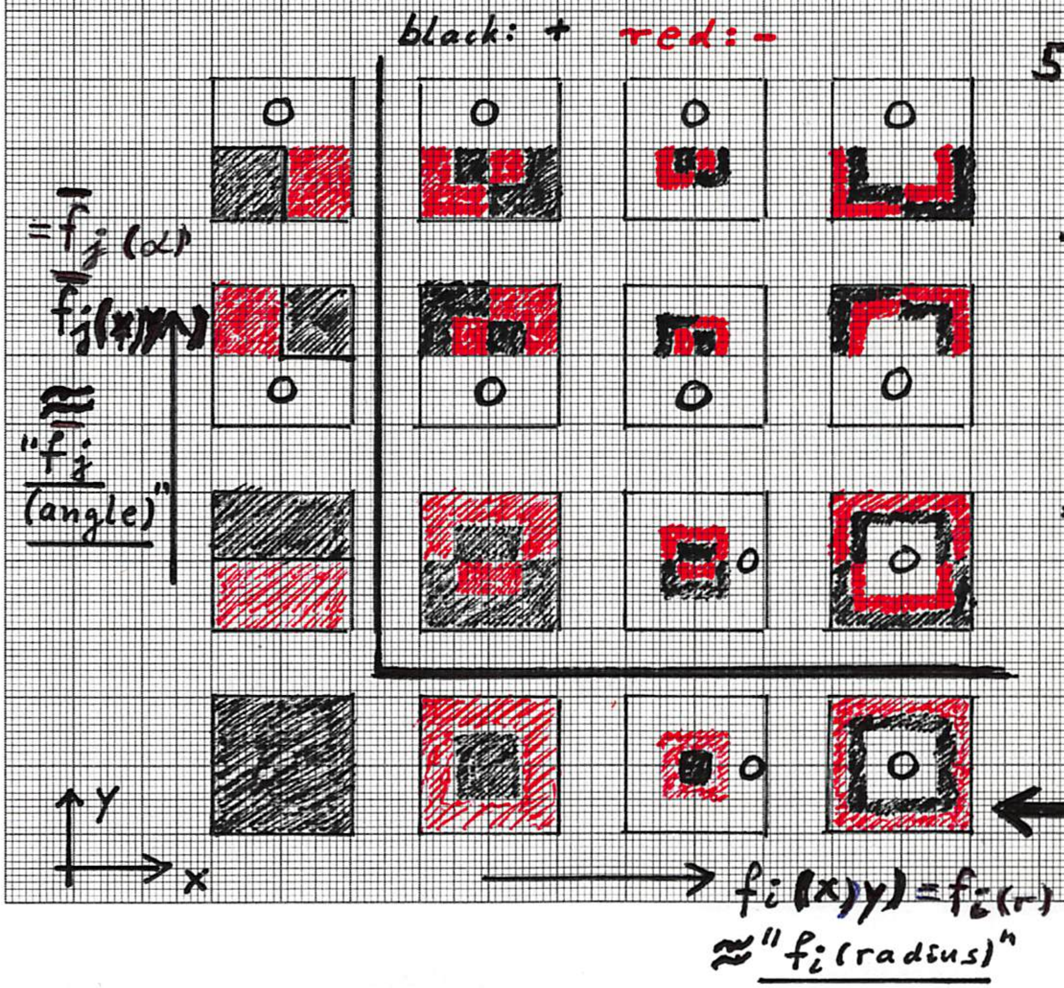
NORMALITY and ORTHOGONALITY!

HAAR WAVELETS – Cont'd.

→ The c_i-values define locally a "MULTI-SCALE SPECTRUM". Using all values for all local neighborhoods in an entire segment defines histograms for the values of $c_0, \ldots, c_{15}$ characterizing the segment in a multi-scale way!

→ Use change-of-variables theorem to compute inner products (integrals) $\int f_i \cdot f_j \, dx\,dy$.

→ Use analytically EXACT or NUMERICAL METHOD to compute $\langle f_i, f_j \rangle$ and $\langle F, f_i \rangle$.



5) HAAR WAVELETS FOR DISK

→ Define basis functions $f_i(x,y)$ (radius) and $\bar{f}_j(x,y)$ (angle)

→ Define tensor product basis functions $f_{i,j}(x,y) = f_i(x,y)\bar{f}_j(x,y)$

← SQUARES REPRESENT DISK DOMAINS:

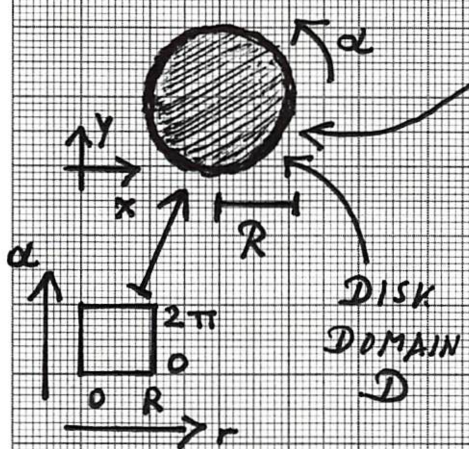
$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x(r,\alpha) \\ y(r,\alpha) \end{pmatrix}$

HAAR WAVELETS – Cont'd.

Examples: Normalization and Orthogonalization of these tensor product basis function

[→ Consider GRAM-SCHMIDT orthogonalization.]

- Use the actual DISK domains:



$f_{00}(x,y) = c_{00}$ (constant)

→ normalization:

$$\| f_{00} \| = \sqrt{\int_D c_{00}^2 \, dA} \quad \leftarrow \text{area}$$

$$= c_{00} \sqrt{\int_D 1 \, dA}$$

$$= c_{00} \pi R^2 = 1$$

$$\Rightarrow c_{00} = \frac{1}{\pi R^2}$$

$$f_{10}(x,y) = \begin{cases} k\, c_{10} & \text{, inner BLACK region} \\ -c_{10} & \text{, outer RED "} \end{cases}$$

→ orthogonalization (with f_{00}):

$$\langle f_{10}, f_{00} \rangle = \int_{BLACK} k\, c_{10} \, dA + \int_{RED} -c_{10} \, dA$$

$$= k\, c_{10} \int_{BLACK} 1 \, dA - c_{10} \int_{RED} 1 \, dA$$

$$= \ldots = c_{10} \pi \left(\frac{R}{2}\right)^2 (k-3) \overset{!}{=} 0$$

$$\Rightarrow k = 3$$

■ HAAR WAVELETS – Cont'd.

... $\Rightarrow$ normalization (of f_{10}):

$$\|f_{10}\| = \sqrt{\langle f_{10}, f_{10}\rangle} = \sqrt{\int_{BLACK \cup RED} (f_{10})^2 \, dA}$$

$$= \sqrt{\int_{BLACK} (3c_{10})^2 \, dA + \int_{RED} (-c_{10})^2 \, dA}$$

$$= \sqrt{9 c_{10}^2 \pi \left(\frac{R}{2}\right)^2 + c_{10}^2 \pi \, 3 \left(\frac{R}{2}\right)^2}$$

$$= \sqrt{12 c_{10}^2 \pi \left(\frac{R}{2}\right)^2} = c_{10} R \sqrt{3\pi} \overset{!}{=} 1$$

$$\Rightarrow c_{10} = \frac{1}{R\sqrt{3\pi}}$$

$\Rightarrow$ f_{00} and f_{10} are orthonormal.

■ HAAR WAVELET COEFFICIENTS (FOR DISK) ENABLE A MULTI-SCALE CHARACTERIZATION OF IMAGE TEXTURE, INVARIANT UNDER TRANSLATION, ROTATION, SCALING.

$\Updownarrow$

- USE GRAM-SCHMIDT ORTHOGONALIZION, possibly slightly adapted to polar coordinate setting and desired symmetry properties of basis functions $f_{i,j}$, TO DEFINE AN ORTHOGONAL (AND NORMALIZED) BASIS!

$\Rightarrow$ BEST APPROXIMATION OF $F(x,y)$:

$$\boxed{A(x,y) = \sum_{j=0}^{3} \sum_{i=0}^{3} a_{i,j} f_{i,j}(x,y), \quad a_{i,j} = \langle F, f_{i,j}\rangle}$$

● $\Rightarrow$ MULTI-SCALE FEATURE VECTOR: $\{a_{i,j}\}$

$\Rightarrow$ minimal computational complexity! ~BH