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9/30/2019

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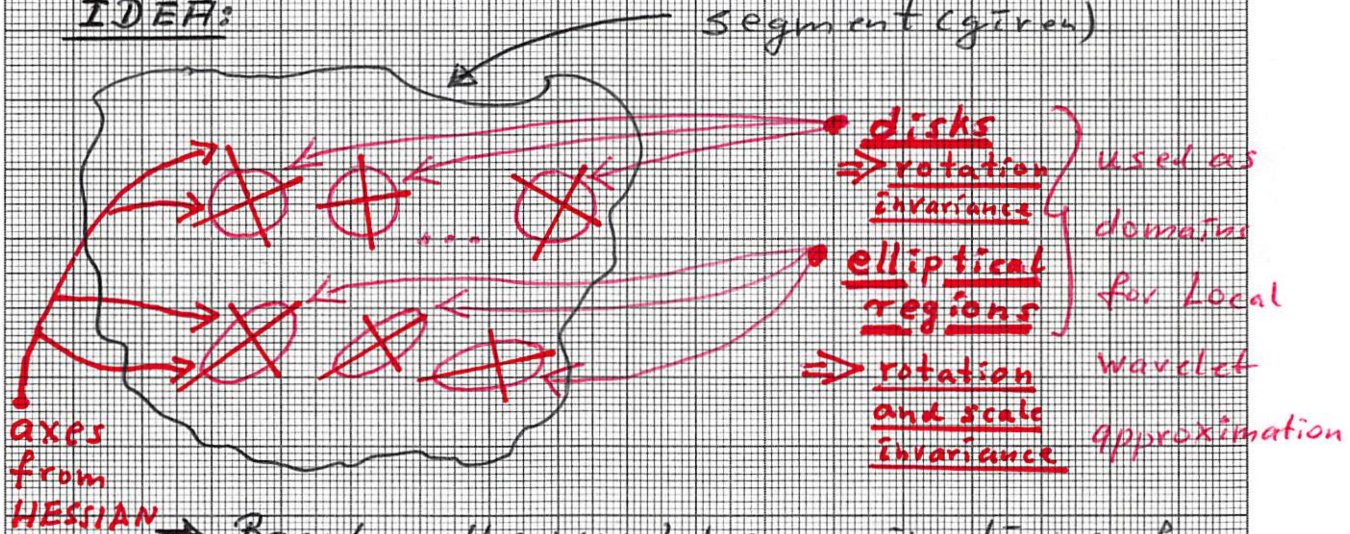
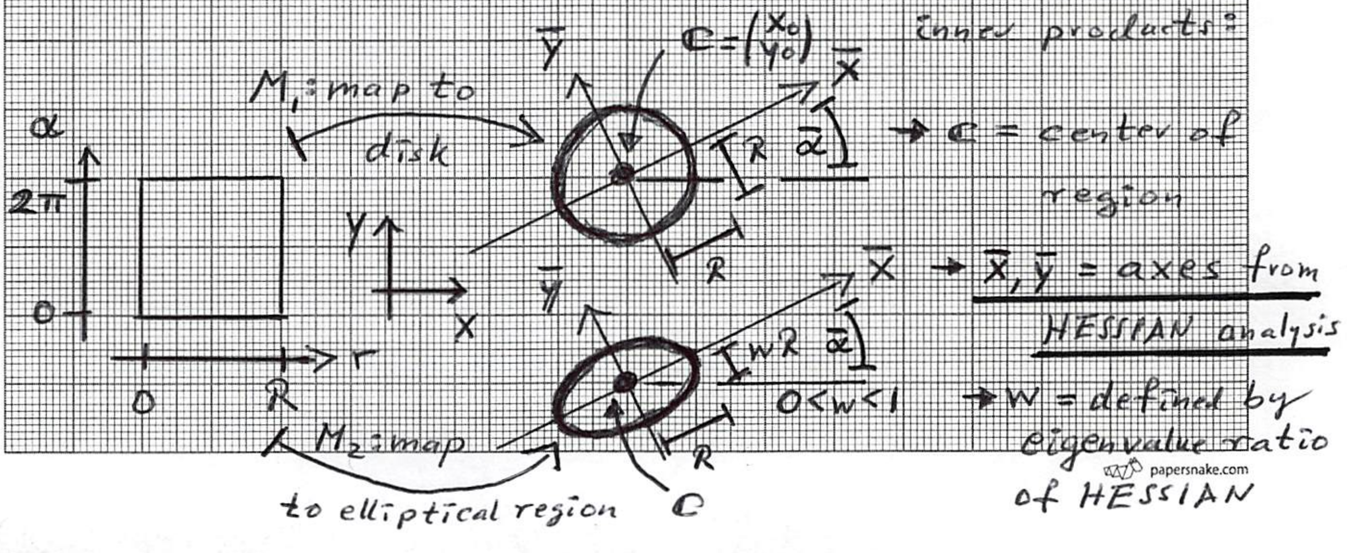
■ HAAR WAVELETS FOR DISKS / ELLIPTICAL REGIONS (BALLS / ELLIPSOIDAL REGIONS) FOR MULTI-RESOLUTION CHARACTERIZATION...

IDEA:



→ Based on the wavelet approximations of all these local sub-domains, establish histograms / distributions of the values of coefficients of all wavelet approximations (⇒ resulting in a SMALL-SCALE and LARGE-SCALE characterization).

1) Parametrization of disks and elliptical regions & change-of-variables theorem... Needed to compute inner products:



➡ Mappings from (r,α)-space to (x,y)-space:

$$M_1: \begin{pmatrix} r \\ \alpha \end{pmatrix} \longmapsto \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \begin{pmatrix} r\cos(\alpha+\bar{\alpha}) \\ r\sin(\alpha+\bar{\alpha}) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \begin{pmatrix} r\,c(\alpha+\bar{\alpha}) \\ r\,s(\alpha+\bar{\alpha}) \end{pmatrix}$$

$\Rightarrow$ absolute value of Jacobian:

$$\underline{|J|} = \left| \det \begin{pmatrix} c(\alpha+\bar{\alpha}) & -r\,s(\alpha+\bar{\alpha}) \\ s(\alpha+\bar{\alpha}) & r\,c(\alpha+\bar{\alpha}) \end{pmatrix} \right| = |r| = \underline{r}$$

$$M_2: \begin{pmatrix} r \\ \alpha \end{pmatrix} \longmapsto \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \begin{pmatrix} r\,c(\alpha+\bar{\alpha}) \\ wr\,s(\alpha+\bar{\alpha}) \end{pmatrix}$$

$$\Rightarrow \underline{|J|} = \left| \det \begin{pmatrix} c(\alpha+\bar{\alpha}) & -r\,s(\alpha+\bar{\alpha}) \\ w\,s(\alpha+\bar{\alpha}) & wr\,c(\alpha+\bar{\alpha}) \end{pmatrix} \right| = \underline{wr}$$

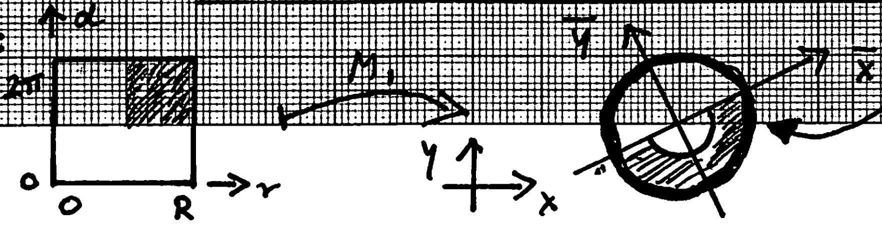
$\Rightarrow$ Change-of-variables for inner products:

$$M_1\text{ (disk)}: \int_{y=\ldots}^{\ldots}\int_{x=\ldots}^{\ldots} f(x,y)\,dx\,dy = \int_{\alpha=\ldots}^{\ldots}\int_{r=\ldots}^{\ldots} r \cdot f(x(r,\alpha), y(r,\alpha))\,dr\,d\alpha$$

M_2 (elliptical region):

$$\int_{y=\ldots}^{\ldots}\int_{x=\ldots}^{\ldots} f(x,y)\,dx\,dy = \int_{\alpha=\ldots}^{\ldots}\int_{r=\ldots}^{\ldots} wr\,f(x(r,\alpha), y(r,\alpha))\,dr\,d\alpha$$

e.g.:



Integration over (x,y)-region reduced to integration over (r,α)-region...

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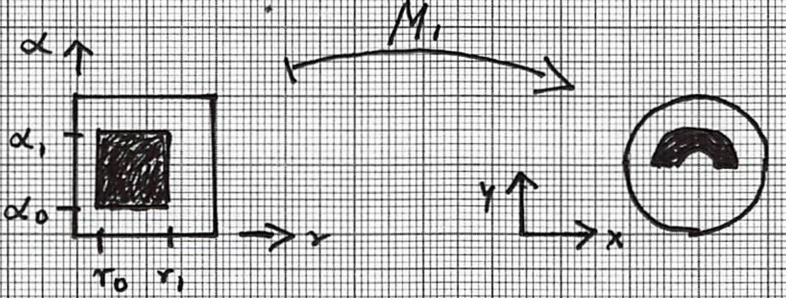
■ HAAR WAVELETS – Cont'd.

Ex: Haar wavelets for DISK domain – needed inner products of constant functions:

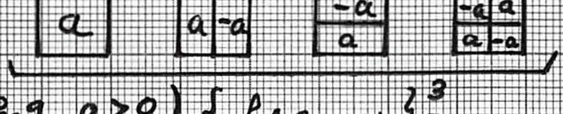


- Basic "building block"/ computation: inner products of two constant functions defined over black-shaded regions – having values c and $\bar{c}$ over ■ in (r,α)-space and C and $\bar{C}$ over ◠ in (x,y)-space, respectively

$$\int_{\alpha_0}^{\alpha_1}\int_{r_0}^{r_1} c\,\bar{c}\,dr\,d\alpha = \underline{c\,\bar{c}\,(r_1-r_0)(\alpha_1-\alpha_0)}$$

$$\int_{◠} C\,\bar{C}\,dx\,dy = \int_{\alpha_0}^{\alpha_1}\int_{r_0}^{r_1} r\,C\,\bar{C}\,dr\,d\alpha$$

$$= \underline{\frac{C\,\bar{C}}{2}\,(r_1^2-r_0^2)(\alpha_1-\alpha_0)}$$



(e.g., $a>0$) $\{f_i(r,\alpha)\}_{i=0}^{3}$

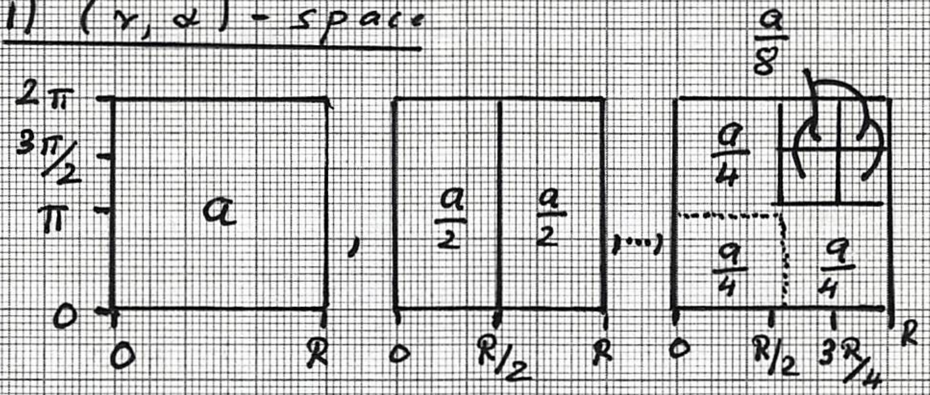
(e.g., $A>0$) $\{F_i(x,y)\}_{i=0}^{3}$

- **ORTHONORMAL BASIS**: $\langle f_i, f_j\rangle = \delta_{i,j}$, $\langle F_i, F_j\rangle = \delta_{i,j}$ ⇒ best approx. = $\sum_i \langle$image, fct$_i\rangle\cdot$fct$_i$

■ HAAR WAVELETS – Cont'd.

➜ Sub-domain areas needed for basis functions:

1) (r, α)-space



e.g., 16 basis functions, $f_0, \ldots, f_{15}$: a = area of original domain

original domain

sub-domains resulting from dyadic splitting

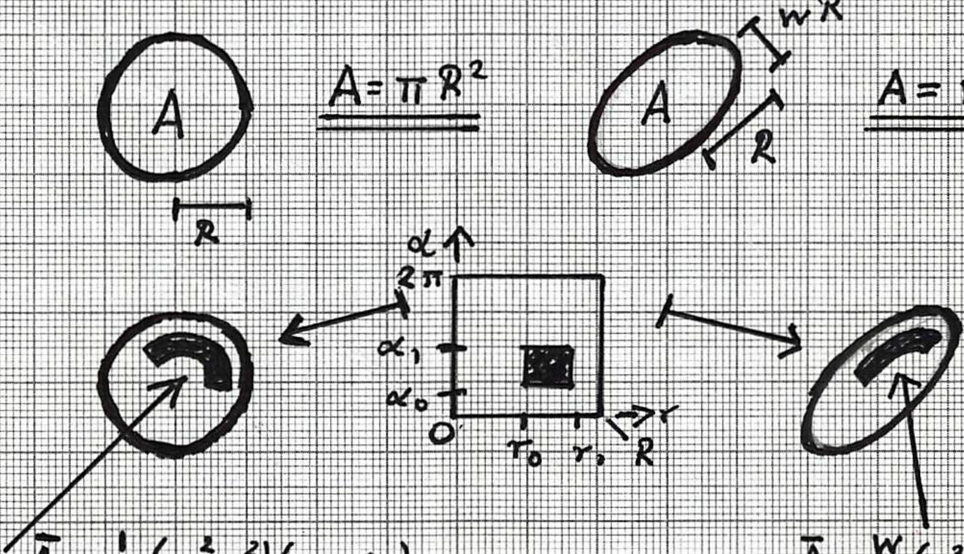
$$\boxed{a = 2\pi R}$$

2) (x, y)-space

i) Disk

ii) Elliptical region ⊛

⊛ NOTE: A wavelet analysis based on elliptical regions can produce (nearly) the same signature for a texture when scaled!



$A = \pi R^2$

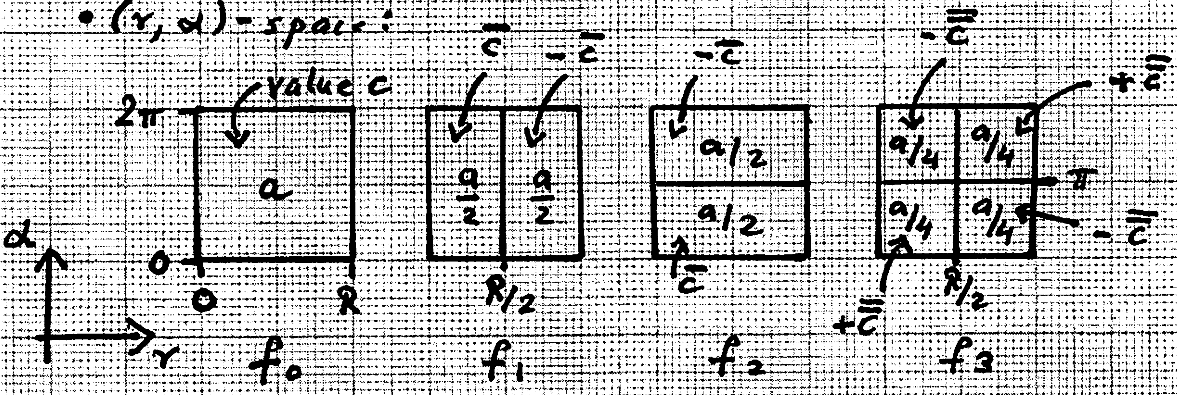
$A = w \pi R^2$

$\bar{A} = \frac{1}{2}(r_1^2 - r_0^2)(\alpha_1 - \alpha_0)$

$\bar{A} = \frac{W}{2}(r_1^2 - r_0^2)(\alpha_1 - \alpha_0)$

HAAR WAVELETS – Cont'd.

- → **ORTHONORMAL wavelet basis functions**
- **Ex:** 4 basis functions for disk (HAAR!)
- (r, α)-space:



$$\|f_0\| = \sqrt{\int_0^{2\pi}\int_0^R c^2\, dr\, d\alpha} = c\sqrt{2\pi R} = c\sqrt{a}$$

(c = positive value, a = area)

$$\Rightarrow c\sqrt{a} = 1 \Rightarrow \underline{\underline{c = 1/\sqrt{a}}}$$

$$\|f_1\| = \sqrt{\int_0^{2\pi}\int_0^{R/2} \bar{c}^2\, dr\, d\alpha + \int_0^{2\pi}\int_{R/2}^{R} (-\bar{c})^2\, dr\, d\alpha}$$

$$= \ldots\ \bar{c}\sqrt{2\pi R} = \bar{c}\sqrt{a} \overset{!}{=} 1 \Rightarrow \underline{\underline{\bar{c} = 1/\sqrt{a}}}$$

$$\|f_2\| = \ldots \qquad \Rightarrow \underline{\underline{\bar{c} = 1/\sqrt{a}}}$$

$$\|f_3\| = \sqrt{\int_0^{\pi}\int_0^{R/2} \bar{\bar{c}}^2\, dr\, d\alpha + \ldots + \int_{\pi}^{2\pi}\int_{R/2}^{R} (-\bar{\bar{c}})^2\, dr\, d\alpha}$$

$$= \sqrt{\ldots} = \bar{\bar{c}}\sqrt{2\pi R} = \bar{\bar{c}}\sqrt{a} \overset{!}{=} 1 \Rightarrow \underline{\underline{\bar{\bar{c}} = 1/\sqrt{a}}}$$

FURTHER: $\underline{\langle f_i, f_j\rangle = \delta_{i,j}} \Rightarrow \underline{\{f_i\}_{i=0}^{3}\ \text{ORTHONORMAL BASIS}}$

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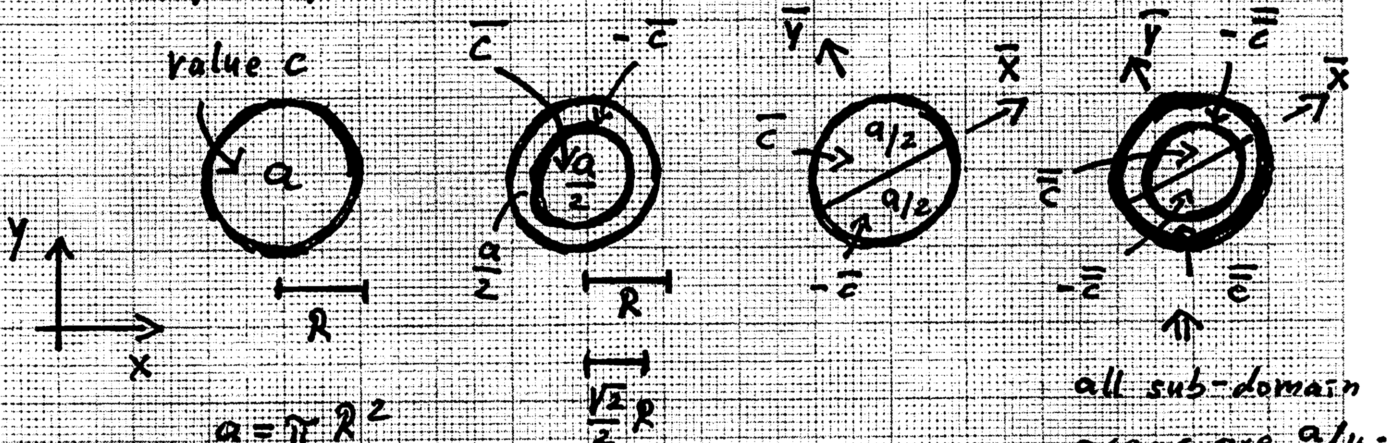
▪ HAAR WAVELETS – Cont'd

- (x, y) - space (disk):



$$\underline{\underline{a = \pi R^2}}$$

all sub-domain areas are a/4.

f_0 — f_1 — f_2 — f_3

$$\|f_0\| = \sqrt{\iint_{\bigcirc} c^2\,dx\,dy} = c\sqrt{\iint_{\bigcirc} dx\,dy} = c\sqrt{\underbrace{\pi R^2}_{=a}} = cR\sqrt{\pi} \overset{!}{=} 1$$

$$\Rightarrow \underline{\underline{c = 1/R\sqrt{\pi} \overset{!}{=} 1/\sqrt{a}}}$$

$$\|f_1\| = \sqrt{\iint_{\bullet} \bar{c}^2\,dx\,dy + \iint_{\circledcirc} (-\bar{c})^2\,dx\,dy} = \sqrt{\bar{c}^2\left(\frac{a}{2} + \frac{a}{2}\right)}$$

$$= \bar{c}\sqrt{a} \overset{!}{=} 1 \Rightarrow \underline{\underline{\bar{c} = 1/\sqrt{a}}}$$

$$\|f_2\| = \sqrt{\ldots} \Rightarrow \underline{\underline{\bar{c} = 1/\sqrt{a}}}$$

$$\|f_3\| = \sqrt{\iint \bar{\bar{c}}^2\,dx\,dy + \ldots + \iint (-\bar{\bar{c}})^2\,dx\,dy} = \bar{\bar{c}}\sqrt{a} \overset{!}{=} 1$$

$$\Rightarrow \underline{\underline{\bar{\bar{c}} = 1/\sqrt{a}}}$$

⇒ **NORMALIZED** FUNCTIONS: $\underline{\|f_i\| = 1}$

- **NOTE:** Can use GRAM-SCHMIDT algorithm to compute ORTHONORMAL basis!

■ HAAR WAVELETS – Cont'd.

⇒ Show that the disk wavelet basis functions are mutually ORTOGONAL:

$$\langle f_0, f_1 \rangle = \int_{\bigcirc} f_0 f_1 \, dx\,dy = \int_{\bullet} c^2 dx\,dy + \int_{\text{◎}} -c^2 dx\,dy = 0$$

$$\langle f_0, f_2 \rangle = \int_{\bigcirc} f_0 f_2 \, dx\,dy = \int_{\text{◓}} c^2 dx\,dy + \int_{\text{◒}} -c^2 dx\,dy = 0$$

$$\langle f_0, f_3 \rangle = \int_{\bigcirc} f_0 f_3 \, dx\,dy = \int_{\text{◓}} c^2 dx\,dy + \int_{\text{◒}} -c^2 dx\,dy$$

$$+ \int_{\text{◠}} -c^2 dx\,dy + \int_{\text{◡}} c^2 dx\,dy = 0$$

$$\langle f_1, f_2 \rangle = \ldots = \int_{\text{◓}} c^2 dx\,dy + \int_{\text{◒}} -c^2 dx\,dy + \int_{\text{◠}} -c^2 dx\,dy + \int_{\text{◡}} c^2 dx\,dy = 0$$

$$\langle f_1, f_3 \rangle = \ldots = \int_{\text{◓}} c^2 dx\,dy + \int_{\text{◒}} -c^2 dx\,dy + \int_{\text{◠}} c^2 dx\,dy + \int_{\text{◡}} -c^2 dx\,dy = 0$$

$$\langle f_2, f_3 \rangle = \ldots \int_{\text{◓}} c^2 dx\,dy + \int_{\text{◒}} c^2 dx\,dy + \int_{\text{◠}} -c^2 dx\,dy + \int_{\text{◡}} -c^2 dx\,dy = 0$$

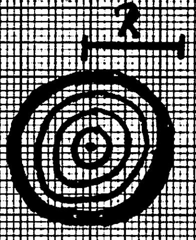
$$\Rightarrow \langle f_i, f_j \rangle = \delta_{i,j} \Rightarrow \underline{\{f_i\}_{i=0}^{3}} \text{ ORTONORMAL BASIS}$$

- **NOTE:** CONSTRUCT ORTHOGONAL BASES WITH MORE BASIS FUNCTIONS FOR DISK AND ELLIPTICAL REGIONS ANALOGOUSLY!

HAAR WAVELETS - Cont'd.

- Needed: Sub-domains in disk (elliptical region) of equal size

i) Disk splitting



R = radius of disk

K = no. of sub-domains

$\Rightarrow$ Disk area = πR^2

Ex.: K = 4

radii: $r_0, \ldots, r_4$

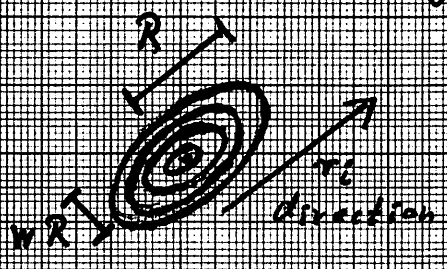
$$\boxed{r_i = \sqrt{\frac{i}{k}} \cdot R} \quad i = 0, \ldots, K$$

Proof: Area of any sub-domain =

$$\frac{1}{2}\left(r_{i+1}^2 - r_i^2\right)(\alpha_1 - \alpha_0) = \frac{1}{2}\left(\frac{i+1}{k}R^2 - \frac{i}{k}R^2\right)2\pi$$

$$= \pi R^2 / K$$

ii) Elliptical region splitting



R = larger radius of region

K = no. of sub-domains

$\Rightarrow$ Region area = $w\pi R^2$

Ex.: K = 4

radii $r_0, \ldots, r_4$

$$\boxed{r_i = \sqrt{\frac{i}{k}} \cdot R} \quad i = 0, \ldots, K$$

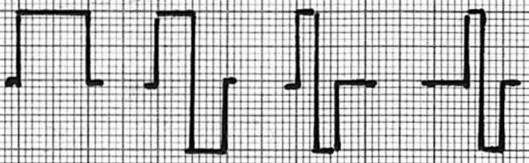
Proof: Sub-domain area =

$= w\pi R^2 / K$

■ HAAR WAVELETS - Cont'd.

➜ Computation of coefficients of wavelet expansion (= best approximation):

- 1D illustration of basis functions (used to define 2D tensor product functions) ⇒



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here: $N=4$

- (r,α)-space tensor product basis functions:

$$f_{i,j}(r,\alpha) = f_i(r)\cdot f_j(\alpha)\,,\quad i,j=0,\dots,3$$

⇒ 16 ($=N^2$) functions

- SINGLE-INDEX notation:

$$f_I(r,\alpha) = f_{N\cdot j+i}(r,\alpha)\,,$$

i.e., $I = N\cdot j+i$, $I=0,\dots,N^2-1$

$(f_{0,0} \rightarrow f_0, \dots, f_{3,3} \rightarrow f_{15})$

⇐

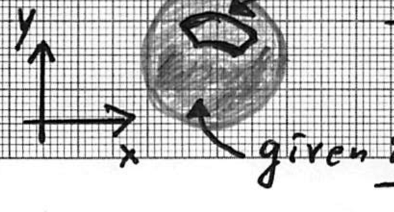
- Best approximation / best wavelet expansion:

$$\text{Approx} = A = \sum_{I=0}^{N^2-1} c_I f_I\,,$$

where $c_I = \langle F, f_I \rangle = \int_{\text{disk}} F\cdot f_I \, dx\,dy = \dots$

given data/image

- Coefficients c_I computed directly in (x,y)-space or (r,α)-space (via change-of-variables theorem):



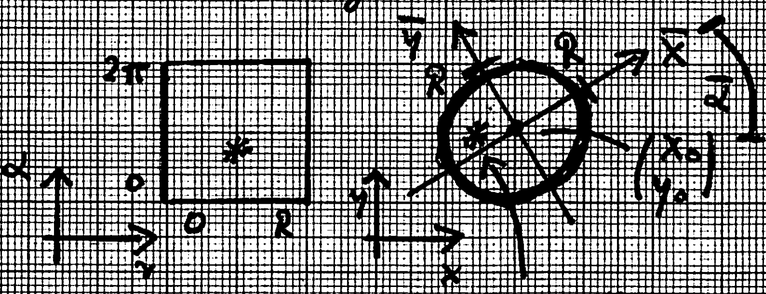
sub-domain in disk for which $\langle F, f_I\rangle$ must be computed

given image F

⇒ Map complicated  region in (x,y)-space to simple rectangle in (r,α)-space

* HAAR WAVELETS - Cont'd.

- Need to map from (r, α)-space to (x, y)-space AND from (x, y)-space to (r, α)-space to use change-of-variables theorem:



$\| * - \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \| = r$

mapping of * :

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \begin{pmatrix} r \cos(\alpha + \bar{\alpha}) \\ r \sin(\alpha + \bar{\alpha}) \end{pmatrix}$$

$\Rightarrow$ must invert mapping!

→ Given point $\begin{pmatrix} x \\ y \end{pmatrix}$, what are the values for r, α?

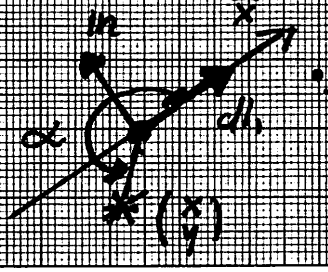
i) $r = \| \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \|$

ii) $\cos(\hat{\alpha}) = \langle d_1, v \rangle$,

where: d_1 = normalized basis vector for $\bar{x}$-axis

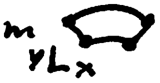
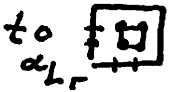
$$v = \left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \right) \Big/ \left\| \left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \right) \right\|$$

$$\Rightarrow \hat{\alpha} = \cos^{-1}(\langle d_1, v \rangle)$$



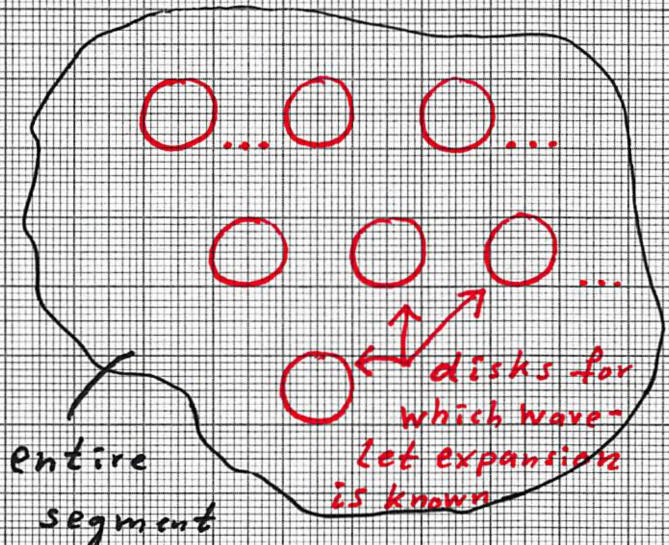
- Define $d_1 = \begin{pmatrix} dx \\ dy \end{pmatrix}$, normal $n = \begin{pmatrix} -dy \\ dx \end{pmatrix}$

$$\Rightarrow \underline{\alpha} = \begin{cases} \hat{\alpha} & \text{if } \langle n, v \rangle \geq 0 \\ 2\pi - \hat{\alpha} & \text{if } \langle n, v \rangle < 0 \end{cases}$$

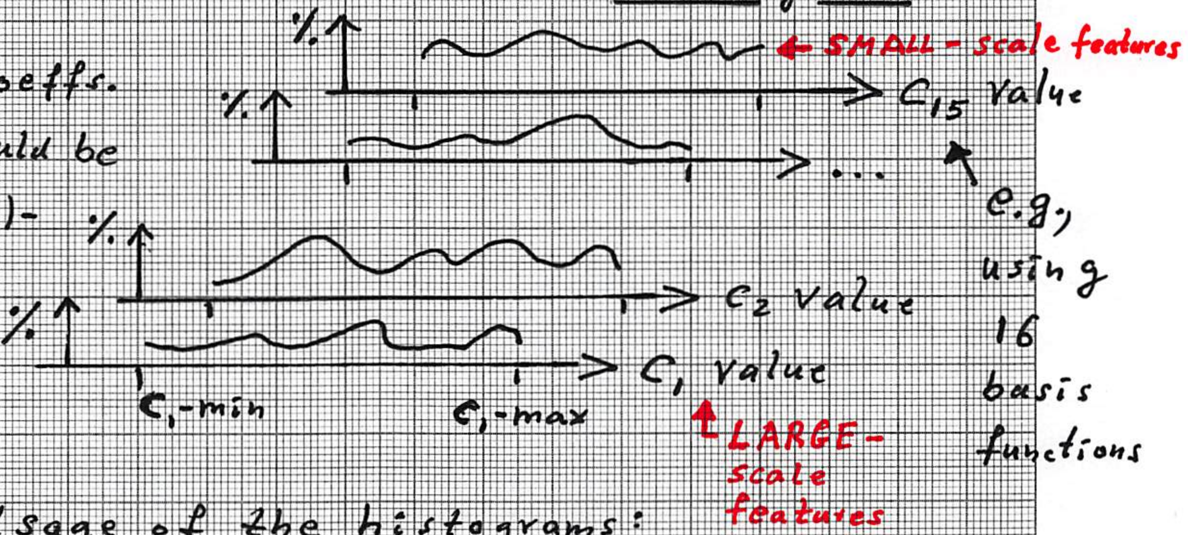
$\Rightarrow$ Map image from  to  and integrate in (r, α)-space. ∴ BH

■ HAAR WAVELETS – Cont'd.

➜ Wavelet basis functions' coefficients C_I define histograms / distributions of C_I-values (considering an entire segment) – thus defining SCALE-/TEXTURE-SPECIFIC SIGNATURES:



- Binning the value ranges of all C_I values and considering all disks used to locally optimally approximate a segment's texture, produces histograms:



- NOTE: Coeffs. C_I could be for (x,y)- or (r,α)-space.

➜ Usage of the histograms:

i) Use histogram of ONE coeff. C_I
⇒ coefficient-/scale-specific analysis

ii) Use histograms of ALL coeffs. C_I
⇒ overall, 'averaged' texture analysis ~BH

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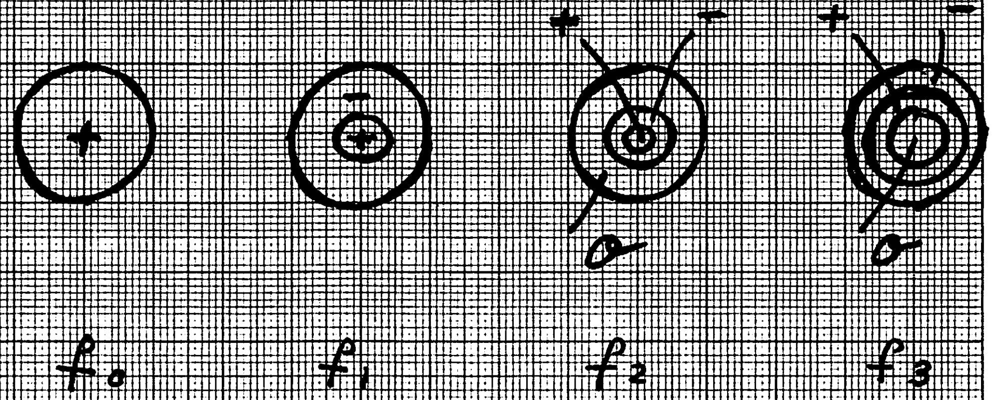
• HAAR WAVELETS - Cont'd.

➜ Construction of (ultimately ORTHONORMAL) wavelet basis, considering RADIUS- and ANGLE-based functions as 'building blocks':

i) Radius-based functions:



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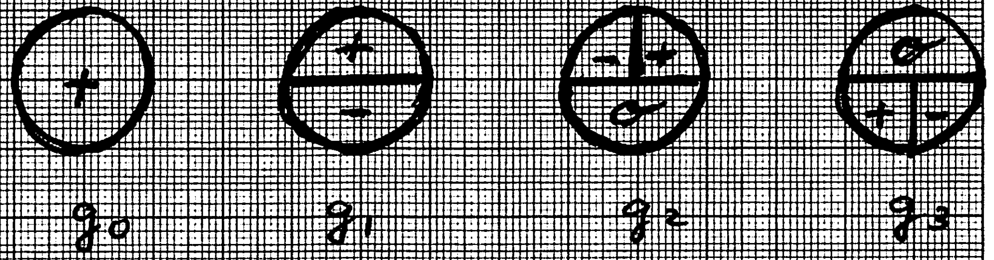
ii) Angle-based functions:



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➜ iii) Resulting tensor products can be used to define ORTHONORMAL bivariate basis functions:

$$h_{0,0}(x,y) = f_0 \cdot g_0, \ldots, h_{3,3}(x,y) = f_3 \cdot g_3 .$$

• PROOF MUST SHOW: $f_{i_1} \perp f_{i_2} \wedge g_{j_1} \perp g_{j_2} \Rightarrow f_{i_1} g_{j_1} \perp f_{i_2} g_{j_2}$

~BH

• NEXT: GENERALIZATION TO 3D VOLUMETRIC CASE!