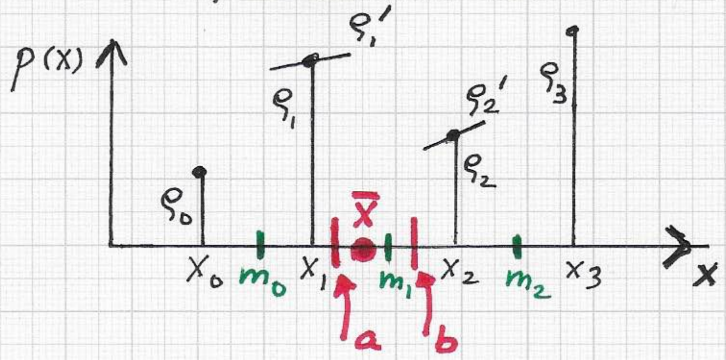
■ DISTRIBUTION APPROXIMATION – Cont'd.

➔ Considerations...

12) C'-/Gradient-continuous p-function.
One can construct local linear Taylor approximations at every site $\mathbf{x}_i$ and use Sibson interpolation to interpolate the linear Taylor approximations. The result is an overall C'-/gradient-continuous p-function.

- 1D, univariate example:



x_i given sites
ϱ_i density values at x_i
m_i midpoints
$\bar{x}$ evaluation location
ϱ_i' derivative (estimate) at x_i
a, b midpoints obtained when "inserting" $\bar{x}$

- Taylor approximations:

$$p_1(x) = \varrho_1 + \varrho_1'(x - x_1)$$

$$p_2(x) = \varrho_2 + \varrho_2'(x - x_2)$$

➔ LOCAL LEAST SQUARES APPOX. NEEDED FOR ϱ_i'

- Sibson interpolation:

$$p(\bar{x}) = \frac{1}{b-a}\Big((m_1 - a)\, p_1(\bar{x}) + (b - m_1)\, p_2(\bar{x}) \Big)$$

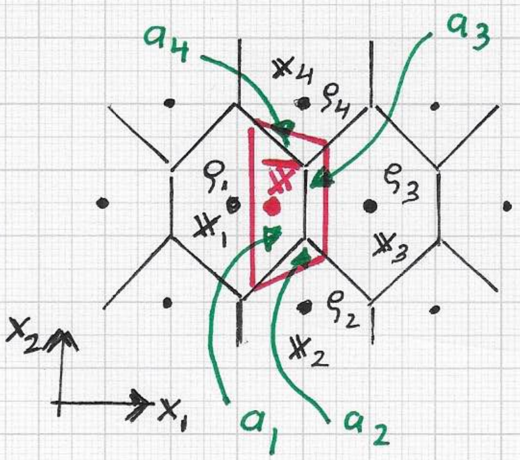
$$= \frac{m_1 - a}{b-a}\, p_1(\bar{x}) + \frac{b - m_1}{b-a}\, p_2(\bar{x})$$

DISTRIBUTION APPROXIMATION – Cont'd.

→ Considerations...

12) ... C^1-/Gradient-continuous ρ-function.

- 2D, bivarate example:



$\mathbb{x}_i = (x_1^i, x_2^i)$ given sites

$\varrho_i = \varrho(\mathbb{x}_i)$ density values

$\bar{\mathbb{x}}$ evaluation loc.

a_i sub-area of tile of site $\mathbb{x}_i$ "cut away" when "inserting" $\bar{\mathbb{x}}$

- Taylor approximation at $\mathbb{x}_i$:

$$p_i(\mathbb{x}) = \varrho_i + \nabla_i \cdot (\mathbb{x} - \mathbb{x}_i) \quad , \text{ where}$$

$$\nabla_i = \left(\frac{\partial}{\partial x_1} \varrho , \frac{\partial}{\partial x_2} \varrho \right)_{\mathbb{x} = \mathbb{x}_i} \; ,$$

$$\mathbb{x} - \mathbb{x}_i = \begin{pmatrix} x_1 - x_1^i \\ x_2 - x_2^i \end{pmatrix}$$

- Sibson interpolation (this specific example):

$$p(\bar{\mathbb{x}}) = \sum_i a_i \, p_i(\bar{\mathbb{x}}) \Big/ \sum_i a_i$$

- **NOTE:** When performing the DISCRETE SIBSON interpolation technique of Park et al. one must add the value of $p_i(\bar{\mathbb{x}})$ to the summed up values at raster point $\bar{\mathbb{x}}$.

DISTRIBUTION APPROXIMATION – Cont'd.

→ Considerations...

12) ... C^1-/Gradient-continuous p-functions.

- General case: k-dim. feature space:

$\mathbf{x}_i = (x_1^i, \ldots, x_k^i)^T$ site in feature space

$\varrho_i = \varrho(\mathbf{x}_i)$ density value at $\mathbf{x}_i$

$p_i(\mathbf{x}) = \varrho_i + \nabla_i \cdot (\mathbf{x} - \mathbf{x}_i)$ Linear Taylor approx.

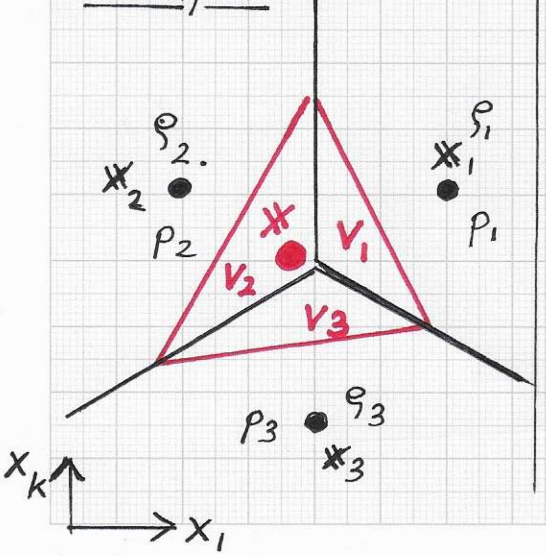
$$\nabla_i = \left(\frac{\partial}{\partial x_1} \varrho, \ldots, \frac{\partial}{\partial x_k} \varrho \right),$$

for $\mathbf{x} = \mathbf{x}_i$,

$$\mathbf{x} - \mathbf{x}_i = \begin{pmatrix} x_1 - x_1^i \\ \ldots \\ x_k - x_k^i \end{pmatrix}.$$

$$\Rightarrow \quad p(\mathbf{x}) = \sum_i v_i\, p_i(\mathbf{x}) \Big/ \sum_i v_i$$

Example:



$\mathbf{x}$ evaluation location

v_i hyper-volumes of sub-regions of Voronoi tiles resulting from "pseudo-insertion" of $\mathbf{x}$ into the Voronoi diagram

$p_i(\mathbf{x})$ Local linear Taylor approximations "blended" to compute $p(\mathbf{x})$ value

~BH