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■ WEIGHTED BEST APPROXIMATION FOR MODELING

→ **Ideas:**
- "Important data" should have more weight.
- "More reliable data" (or "data with less noise") should have more weight.
- In the case of data dependent on an independent variabe X, "data in certain regions of the X-domain" should have more weight when these regions are of specific relevance.

⇒ Consider WEIGHTED best approximation for the computation of a model (= model of feature value distribution of a material) of the feature value probability density function of an object of specific matrrial type.

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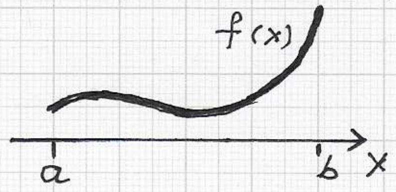
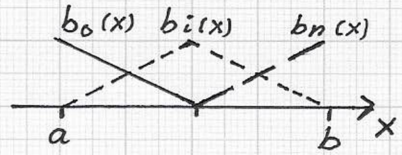
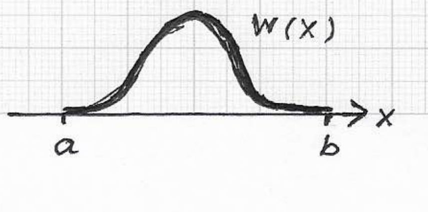
1) Weighted best approximation – CONTINUOUS case







→ given:
- function $f(x)$, $x \in [a,b]$
- basis functions $b_i(x)$, $i = 0 \ldots n$
- weight function $w(x) \geq 0$, $x \in [a,b]$

→ wanted:
- (weighted) best approximation $a(x) = \sum_{i=0}^{n} c_i b_i(x)$

WEIGHTED BEST APPROXIMATION - Cont'd.

... Continuous case...

$$\Rightarrow \text{Minimize} \int_{x=a}^{b} w(x)\,\big(f(x) - a(x)\big)^2\,dx$$

- Notation for inner products:

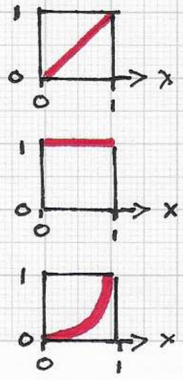
$$\langle b_i(x), b_j(x) \rangle = \int_{x=a}^{b} w(x)\, b_i(x)\, b_j(x)\, dx$$

$$\langle f(x), b_i(x) \rangle = \int_{x=a}^{b} w(x)\, f(x)\, b_i(x)\, dx$$

- Solve NORMAL EQUATIONS to obtain coefficients c_i:

$$\textbf{!}\quad \begin{bmatrix} \langle b_0(x), b_0(x)\rangle & \cdots & \langle b_0(x), b_n(x)\rangle \\ \vdots & & \vdots \\ \langle b_n(x), b_0(x)\rangle & \cdots & \langle b_n(x), b_n(x)\rangle \end{bmatrix} \begin{bmatrix} c_0 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} \langle f(x), b_0(x)\rangle \\ \vdots \\ \langle f(x), b_n(x)\rangle \end{bmatrix}$$

(Use functions $b_i(x)$ with local, compact support. $\Rightarrow$ SPARSE matrix)



$f(x) = x$

$b_0(x) = 1$

$w(x) = x^k$

Example: $\langle b_0(x), b_0(x) \rangle = \int_0^1 x^k \cdot 1 \cdot 1\, dx = \frac{1}{k+1}$

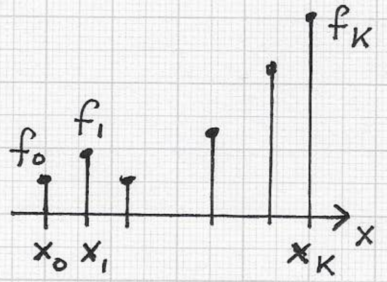
k	c_0
0	1/2
1/1000	1001/2001
1	2/3
1000	1001/2002

$\langle f(x), b_0(x) \rangle = \int_0^1 x^k \cdot x \cdot 1\, dx = \frac{1}{k+2}$

$a(x) = c_0\, b_0(x)$ $\quad \Rightarrow \frac{1}{k+1}\, c_0 = \frac{1}{k+2} \Rightarrow c_0 = \frac{k+1}{k+2}$

■ WEIGHTED BEST APPROXIMATION – Cont'd.

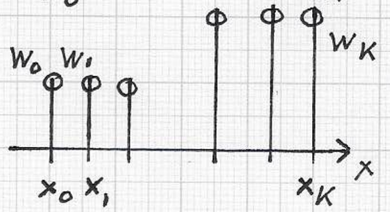
2) Weighted best approximation – DISCRETE case



→ given: • function values $f_k = f(x_k)$, $k=0\ldots K$, $x_0 < x_1 < \ldots < x_K$

• basis functions $b_i(x)$, $i = 0 \ldots n$

• weights w_k, $k=0\ldots K$, $w_k \geq 0$



→ wanted: • (weighted) best approximation $a(x) = \sum_{i=0}^{n} c_i b_i(x)$

$$\Rightarrow \text{Minimize} \sum_{k=0}^{K} w_k \left(f_k - a(x_k)\right)^2$$

• Notation for inner products:

$$\langle b_i(x), b_j(x) \rangle = \sum_{k=0}^{K} w_k \, b_i(x_k) \, b_j(x_k)$$

$$\langle f^{*}, b_i(x) \rangle = \sum_{k=0}^{K} w_k \, f_k \, b_i(x_k)$$

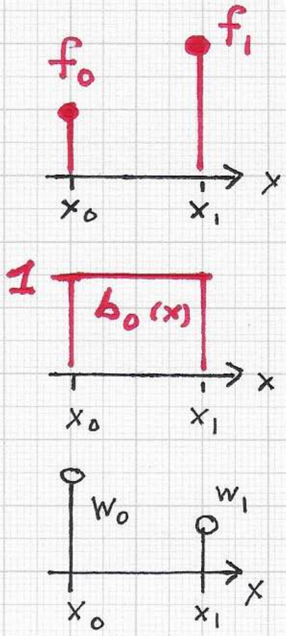
* f denotes the set of samples $f_0 \ldots f_K$.

▪ WEIGHTED BEST APPROXIMATION – Cont'd.

... Discrete case ...

- Solve <u>NORMAL EQUATIONS</u> to obtain coefficients c_i:

$$\begin{bmatrix} \langle b_0(x), b_0(x)\rangle & \cdots & \langle b_0(x), b_n(x)\rangle \\ \vdots & & \\ \langle b_n(x), b_0(x)\rangle & \cdots & \langle b_n(x), b_n(x)\rangle \end{bmatrix} \begin{bmatrix} c_0 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} \langle f, b_0(x)\rangle \\ \vdots \\ \langle f, b_n(x)\rangle \end{bmatrix} \quad !$$



- Example:

$$\langle b_0(x), b_0(x)\rangle = \sum_{k=0}^{1} w_k \cdot 1 \cdot 1$$

$$= w_0 + w_1$$

$$\langle f, b_0(x)\rangle = \sum_{k=0}^{1} w_k f_k b_0(x_k)$$

$$= w_0 f_0 + w_1 f_1$$

$$\Rightarrow (w_0 + w_1)\, c_0 = w_0 f_0 + w_1 f_1$$

$$\hookrightarrow c_0 = \frac{w_0 f_0 + w_1 f_1}{w_0 + w_1}$$

$$a(x) = c_0 b_0(x)$$

→ Often: (i) $\sum_i w_k = 1$ and (ii) $w_k \geq 0$, $k = 0 \ldots K$

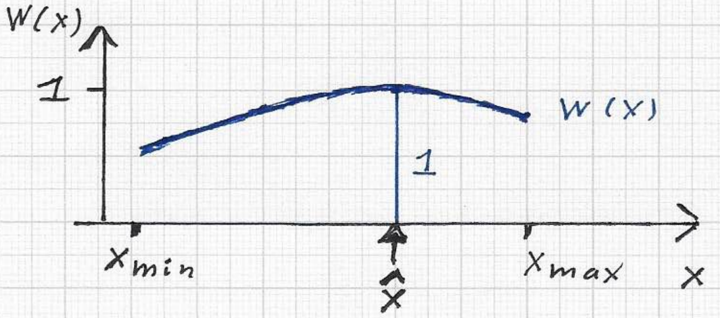
w_0	1	0	1/2	w_0
w_1	0	1	1/2	$1 - w_0$
c_0	f_0	f_1	$(f_0 + f_1)/2$	$w_0 f_0 + (1 - w_0) f_1$

WEIGHTED BEST APPROXIMATION – Cont'd.

➔ Choice of Weight Function $w(x)$ / Weights w_k

- "Important data" ⇒ more weight

Example: Considering the case of a simple univariate distribution function $p(x)$, where the 1D domain space x represents density of a material, assign relatively larger weights to to data closer to the "ideal," "perfect" density $\hat{x}$ of the specific material:



- here: x_{min} (x_{max}) = minimal (maximal) value of measured density x

e.g., $w(x) = \dfrac{1}{1 + (d/h)^2}$, where $d = (x - \hat{x})$, $h = x_{max} - x_{min}$

⇒ (i) Given discrete data, the needed weights w_k for K given values f_k, $k = 0 \ldots K$, could be defined as the values of such a weight function $w(x)$ (with x representing the given function values).

(ii) The 1D domain space x becomes a multi-dimensional domain space $\mathbb{X}$ for multiple features.

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