## OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions:

THREE RELEVANT ISSUES CONCERNING SCALE & RESOLUTION:

1) Scale of pixels/voxels defining a segment

This scale makes possible the analysis of a segment at the smallest scale.

2) (Average) Scale of "texels," texture elements, defining the "discontinuous granular pieces" of a segment

This scale makes possible the analysis of the "granular make-up" of a material made up by distinct regions – regions with individual characteristics and characteristics summarizing their combined statistical behavior.

3) Spectral scales of "eigenfunction convolution masks"

Based on the number of pixels/voxels defining an eigenfunction analysis mask, these spectral scales make possible the (local) characterization of segment regions – characterized individually or in conjunction in a statistical manner.

The simple and "ideal" checkerboard texture/pattern can be viewed as an example of a 2-material class, where "discontinuous texture elements" are either of mass-type $m$ or mass-type $M$ (with $m$ or $M$ being the specific mass values associated with unit-area square pixels).

When a 2D segment is given and must be analyzed, characterized and classified, one can employ a **single-scale or multi-scale analysis** approach. Keeping the checkerboard texture/pattern in mind as a motivating 2-material class, one must determine answers to questions like these:

→ Is it necessary to characterize the segment for pixel-level/-scale behavior?

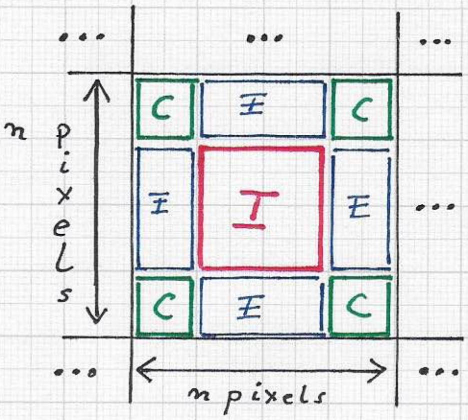
→ Is it sufficient to characterize segment behavior in the form of certain types of averaged behavior? Which average to use?

→ When the segment exhibits "textured behavior," is it necessary to determine, for example, the numbers of "texture elements" (e.g., $m$- and $M$-type elements), their areas, their ratio, distribution in the segment etc.?

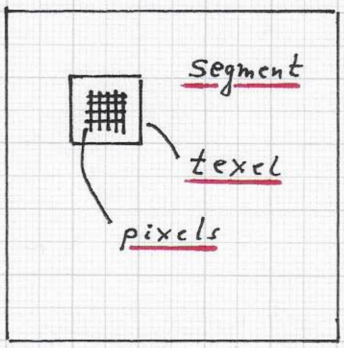
**The classification problem implies the scale(s).**

## ■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions:




One square "texel" to which the 5-pixel mask is applied. Mask center pixels are of types I (interior), E (edge) or C (corner), and the "texel" has a pixel resolution of $n \times n$ pixels. The entire "checkerboard" segment consists of $8 \times 8 = 64$ "texels" with alternating texel values of $m$ and $M$.




Three scales are considered: Segment, texel, pixel. The convolution mask is used at the pixel level, producing a spectral frequency characterization at the pixel scale – which can be "integrated to define characteristics at the texel and segment scales.

We consider a simple and "ideal" case for studying the interplay of the various SCALES described on the previous page: Given is a segment that represents a perfect checkerboard pattern/texture of $8 \times 8 = 64$ unique "texels" with associated values $m$ and $M$ that alternate from texel to texel. Further, we assume that each texel has a pixel resolution of $32 \times 32$ pixels, i.e., the total number of pixels is $1024 \times 1024$. We apply the 5-pixel mask to all pixels and compute its response, using each of the $1024 \times 1024$ segment pixels as mask center pixel.

Of interest are the spectral responses at the pixel level, the texel level and the segment level, keeping in mind that the mask's spectral response provides multi-frequency characteristics.

We recall that the spectral coefficient tuple responses $\mathbb{C}$ of the 5-pixel mask are:

$$\mathbb{C}^{INT} = \sqrt{2}/2\ (m, 0, 0, 0, 3m) \quad \text{or}$$

$$\mathbb{C}^{INT} = \sqrt{2}/2\ (M, 0, 0, 0, 3M) \ ,$$

$$\mathbb{C}^{EDG} = \sqrt{2}/4\ (m+M, 0, 0, 0, 3(m+M)) \ ,$$

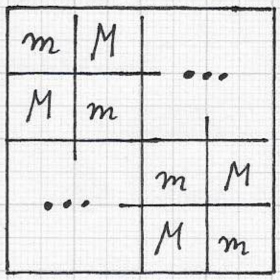
$$\mathbb{C}^{COR} = \sqrt{2}/4\ (m+M, 0, 0, 0, 3(m+M)) \ . \ldots$$

(See pp. 21–23, 11/3/21 – 11/5/21.)

## ■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: ... The tuples $\mathbb{C}^{INT}$, $\mathbb{C}^{EDG}$ and $\mathbb{C}^{COR}$ define the spectral responses for the interior, edge and corner cases to be considered for the mask's center pixel location. For our example, we use the specific values

$$m = 4 \quad \text{and} \quad M = 8 \, .$$




Segment considered in the example: the "texels" are squares with associated alternating values m and M.

We obtain the following pixel-scale responses for the two possible interior cases:

$$\mathbb{C}^{INT}(m) = \sqrt{2}/2\,(m, 0, 0, 0, 3m) = \sqrt{2}\,(2, 0, 0, 0, 6) \, ,$$

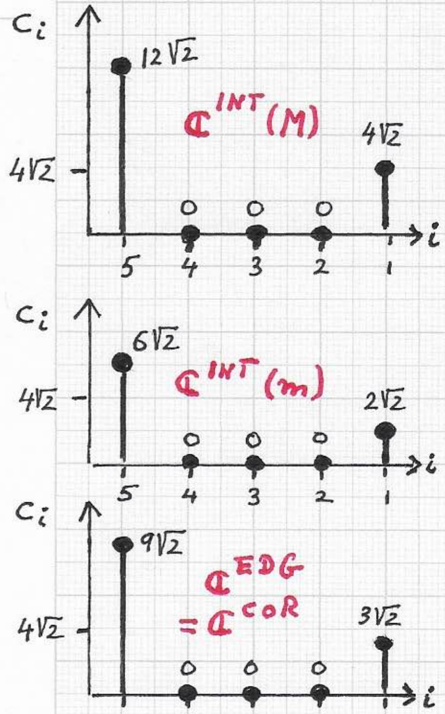
$$\mathbb{C}^{INT}(M) = \sqrt{2}\,(4, 0, 0, 0, 12) \, .$$

The (averaged) pixel-scale response for all the texel "edge pixels" is:

$$\mathbb{C}^{EDG} = \sqrt{2}/4\,(m+M, 0, 0, 0, 3(m+M)) = \sqrt{2}\,(3, 0, 0, 0, 9) \, .$$

Each texel produces four mask responses with mask center pixels being texel corner pixels; the (averaged) pixel response for all the texel "corner pixels" is:

$$\mathbb{C}^{COR} = \sqrt{2}/4\,(m+M, 0, 0, 0, 3(m+M)) = \sqrt{2}\,(3, 0, 0, 0, 9) \, .$$

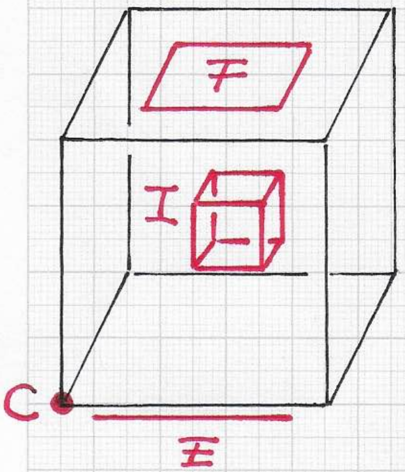



The three spectra for the alternating texel pattern of m- and M-values. Coefficients $c_5$ and $c_1$ capture all information.

[ In order to avoid boundary conditions when processing/analyzing the segment's boundary, we assume that the m-texels and M-texels continue beyond the segment's boundary. ]

## ■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: For a detailed quantitative understanding, we calculate the numbers of $\mathbb{C}^{INT}(m)$, $\mathbb{C}^{INT}(M)$, $\mathbb{C}^{EDG}$ and $\mathbb{C}^{COR}$ cases that arise in our example with **8×8** texels and **32×32** pixels per texel.




**1)** Numbers of $\mathbb{C}^{INT}(m)$ and $\mathbb{C}^{INT}(M)$ cases, $nI = nI(m) = nI(M)$:

$$\mathbf{nI = 8 \cdot 8 \cdot 30 \cdot 30 = 57600}$$

**2)** Number of $\mathbb{C}^{EDG}$ cases, $nE$:

$$\mathbf{nE = 8 \cdot 8 \cdot 4 \cdot 30 = 7680}$$

**3)** Number of $\mathbb{C}^{COR}$ cases, $nC$:

$$\mathbf{nC = 8 \cdot 8 \cdot 4 = 256}$$

Generalization of the 2D checkerboard texture with alternating values $m$ and $M$ to the 3D case: There are $t \cdot t \cdot t = t^3$ volumetric texels, each texel having a resolution of $v \cdot v \cdot v = v^3$ voxels. Thus, the related **7-voxel mask** has center voxels that have one of four possible types, corner type **(C)**, edge type **(E)**, face type **(F)** or interior type **(I)**. The resulting numbers **nC**, **nE**, **nF** and **nI**, respectively, for these mask center voxel types are:

$$\boxed{\begin{aligned} nI &= t^3(v-2)^3 \\ nF &= 6t^3(v-2)^2 \\ nE &= 12t^3(v-2) \\ nC &= 8t^3 \\ \Sigma &= t^3 v^3 \end{aligned}}$$

We obtain more general expressions by using $(t \cdot t) = t^2$ as number of texels and $(p \cdot p) = p^2$ as number of pixels per texel. The resulting more general case numbers are:

$$\boxed{\begin{aligned} nI &= t^2(p-2)^2 \\ nE &= 4t^2(p-2) \\ nC &= 4t^2 \\ nI + nE + nC &= t^2 p^2 \end{aligned}}$$

**!**

• **Note:** The values of these numbers can be written as

$$t^2p^2 = t^2 \cdot ((p-2)+2)^2 = t^2 \cdot ((p-2)^2 + \underline{4}(p-2) + \underline{4}) \text{ and}$$

$$t^3v^3 = t^3 \cdot ((v-2)+2)^3 = t^3 \cdot ((v-2)^3 + \underline{6}(v-2)^2 + \underline{12}(v-2) + \underline{8}).$$

## OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• **Laplacian eigenfunctions:** For a detailed understanding of the spectral responses of the 5-pixel mask and the relevance for image / segment analysis across MULTIPLE SCALES – from pixel-level to segment-level – we provide the mask responses at the pixel-level. Again, we use the checkerboard texture with square texels with alternating values $m$ and $M$ as a simple, ideal case. The table provided below includes all possible cases one must consider.

$$\begin{array}{|c|c|c|} \hline & f_1 & \\ \hline f_2 & f_3 & f_4 \\ \hline & f_5 & \\ \hline \end{array} \quad = \quad \begin{array}{ccc} & f_1 & \\ f_2 & f_3 & f_4 \\ & f_5 & \end{array}$$

Simplified representation of the 5-pixel mask and the 5 associated values.

| | Case | $\mathbb{C} = (c_1, c_2, c_3, c_4, c_5)$ | Average $\mathbb{C}$-tuple of dual cases |
|---|---|---|---|
| INTERIOR | ① $\begin{array}{ccc} & m & \\ m & m & m \\ & m & \end{array}$ | $\sqrt{2}/2\,(m, 0, 0, 0, 3m)$ | $\sqrt{2}/4\,(m+M, 0, 0, 0, 3(m+M))$ |
| | ② $\begin{array}{ccc} & M & \\ M & M & M \\ & M & \end{array}$ | $\sqrt{2}/2\,(M, 0, 0, 0, 3M)$ | |
| VERTICAL EDGE | ③ $\begin{array}{ccc} & m & \\ m & m & M \\ & m & \end{array}$ | $\sqrt{2}/4\,(m+M, m-M, 3(M-m), m-M, 5m+M)$ | $\sqrt{2}/4\,(m+M, 0, 0, 0, 3(m+M))$ |
| | ④ $\begin{array}{ccc} & M & \\ M & M & m \\ & M & \end{array}$ | $\sqrt{2}/4\,(m+M, M-m, 3(m-M), M-m, 5M+m)$ | |
| | ⑤ $\begin{array}{ccc} & M & \\ m & M & M \\ & M & \end{array}$ | $\sqrt{2}/4\,(m+M, M-m, M-m, 3(m-M), 5M+m)$ | $\sqrt{2}/4\,(m+M, 0, 0, 0, 3(m+M))$ |
| | ⑥ $\begin{array}{ccc} & m & \\ M & m & m \\ & m & \end{array}$ | $\sqrt{2}/4\,(m+M, m-M, m-M, 3(M-m), 5m+M)$ | |