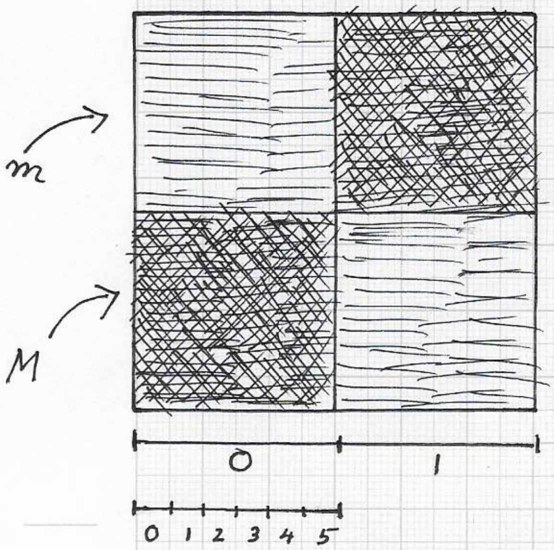
OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: viii) Considering our simple example of a segment of t^2p^2 pixels with values m and M, with t being the number of square texels and p being the number of pixels per texel, one can view a new segment to be classified as a "perfect match" of the alternating m-/M-value checkerboard class when:

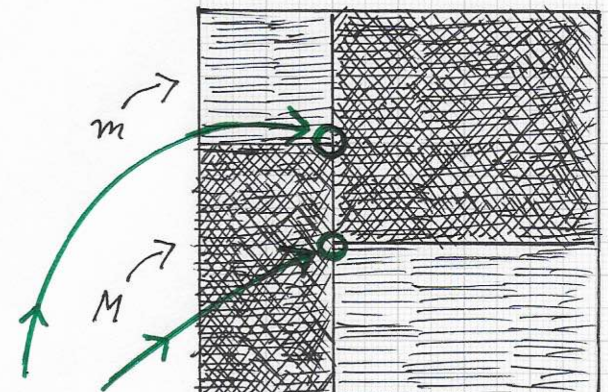
1. All $\mathbb{C}$-tuple values of the segment being analyzed are within small distances of the to-be-detected "ideal" $\mathbb{C}$-tuples.
2. The numbers N_i determined for $\mathbb{C}$-tuple responses viewed as responses for the "ideal" $\mathbb{C}_i$-tuples are equal to the ideal number N_i^{ID}. (The new segment being analyzed also has exactly t^2p^2 total pixels!)

The "counter tuple" $\mathbb{N} = (N_1, \ldots, N_{18})$ depends on the numbers of texels and pixels per texel! $\mathbb{N} = \mathbb{N}(t, p)$. As a consequence, this question arises: **TO WHAT DEGREE DOES THE VALUE OF $\mathbb{N}$ DEPEND ON THE SHAPE, THE GEOMETRY, OF THE TEXELS?**

MORE GENERAL TESSELLATIONS MUST BE CONSIDERED – BEYOND CHECKERBOARDS!



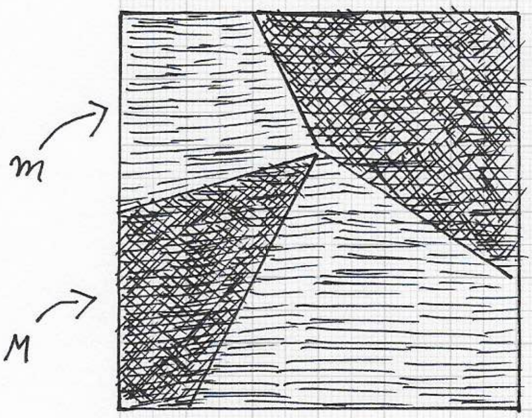
Ideal checkerboard data for which a "perfect" analysis and classification is possible: 2×2 texels, each texel having pixel resolution 6×6, with error-free mass values m or M.



Same square segment/region represented by a 4-texel tessellation: texels have different pixel resolutions; pixels still have error-free mass values m and M.

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: ••• The illustrations on the previous page and this page are sketches of increasingly complex "2-material," (m, M)-value data sets. (An underlying high-resolution pixel grid is not shown here.) The texels of m-type or M-type are shown as lighter or darker regions. As far as a "topological characterization" is concerned, one can interpret/view the texels as "tiles" of a tessellation of the entire data set (segment) to be analyzed. Considering the four illustrated cases, one can say:



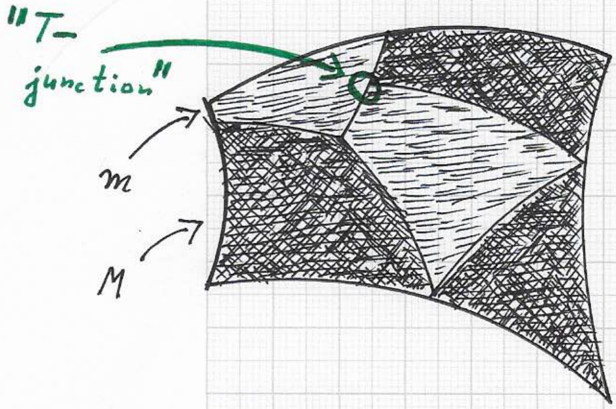
Same square segment/region with non-square texels, each having a different number of pixels. Pixels still have error-free mass values m and M.

→ **A texel is a 2D tile with an associated mass-per-pixel value, having multiple edges (boundaries, boundary curves).**

→ **Two texels are neighbors when they share a common edge.**

(Texels can also share parts of boundary curves, defining "T-junctions.")

→ **Multiple edges of multiple texels can end in a shared point, a "corner vertex."**



More general segment/region with non-square shape. Texels have curved boundaries and different pixel resolutions. The tessellation also contains "T-junctions." Pixels still have error-free mass values m, M.

⇒ Generalization: Pixels have mass values $m^{+}/_{-}\varepsilon$ and $M^{+}/_{-}\varepsilon$; mass values vary within the texels.

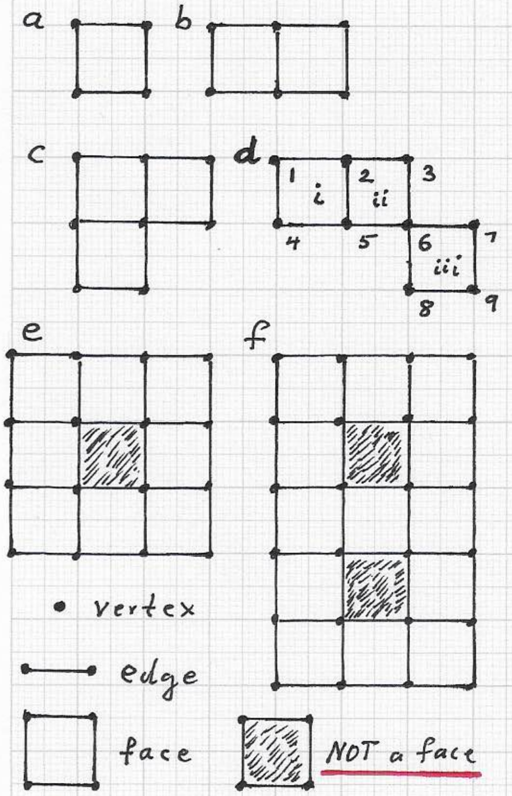
It is possible to adapt methods from computational topology for the purpose of segment analysis and classification.

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: • **Note**: The EULER (or Euler-Poincaré) characteristic χ provides a means to analyze topological aspects of graphs and "**cell complexes**." Considering the examples of 2D image segments shown on this page, χ relates the numbers of vertices, edges and pixels (= faces):

$$\chi = V - E + F .$$

V is the number of pixel corners of a segment; **E** is the number of unique pixel edges connecting pixel corners; and **F** is the number of pixels (or the unique 4-edge closed "loops," "circles," defining pixel boundaries). One must compute and represent the topology of a segment to perform a "χ-analysis."



Six simple **2D segments**, i.e., connected sets of pixels, connected via shared edges or shared vertices. Segments define graphs: "pixel complexes" or "quadrilateral complexes". Vertices **V** are corners of pixels; edges **E** are connections between vertices; faces **F** are "loops," "circles" of four edges defining a cycle.

case	V	E	F	χ
a	4	4	1	1
b	6	7	2	1
c	8	10	3	1
d	9	11	3	1
e	16	24	8	0
f	24	38	13	-1

Associated Euler characteristics.

edge	v1	v2
1	1	2
2	2	3
3	4	5
4	5	6
5	6	7
6	8	9
7	1	4
8	2	5
9	3	6
10	6	8
11	7	9

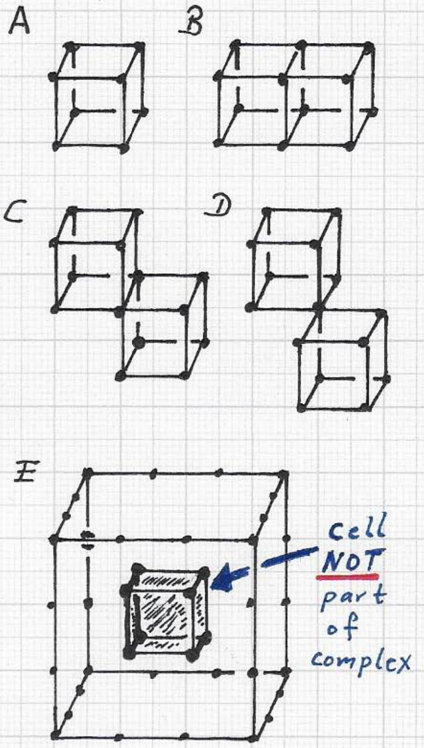
face	edges
i	1, 3, 7, 8
ii	2, 4, 8, 9
iii	5, 6, 10, 11

These "edge" and "face tables" define the topology of case **d**. The "case table" lists the χ values for all six cases.

...

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

Laplacian eigenfunctions: ... (Since our applications only concern 2D or 3D segments defined by sets of pixels (as faces) or voxels (as cells), we do not discuss the more general cases where complexes consist of arbitrary polygonal faces or polyhedral cells, respectively.)



Five simple 3D segments, i.e., connected sets of voxels, connected via shared faces, edges or vertices. The voxels are the cells C of the complex, bounded by six faces. Case E: 3×3×3 voxel neighborhood without the "center voxel."

case	V	E	F	C	χ
A	8	12	6	1	1
B	12	20	11	2	1
C	14	23	12	2	1
D	15	24	12	2	1
E	64	144	108	26	2

Associated Euler characteristics.

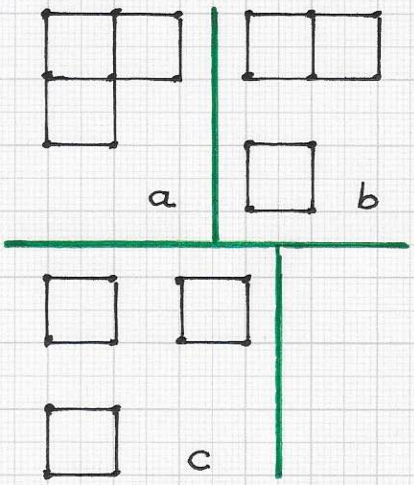
We are primarily interested in segments consisting of voxels, defining the C "cells" of a 3D complex. In this case, the Euler characteristic is defined as

$$\chi = V - E + F - C .$$

Cases A, B, C and D shown in the figure (Left) show voxel segments with Euler characteristic 1, while case E has Euler characteristic 2. Computationally, one has a voxel segment as input; the voxels define the C unique 3D cells; the voxel faces define the F unique 2D faces (with each shared face only counted once); the voxel edges define the E unique 1D edges (with each shared edge only counted once); and the voxel corners define the V unique 0D vertices of the complex (with each shared corner only counted once). ...

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: ... Calculating and storing the topological information needed for the computation of a voxel segment – i.e., cell, face, edge and vertex tables – is computationally expensive. Regardless, it is possible to potentially consider the use of the Euler characteristic of a segment as a viable segment-level feature. The examples shown in the figure (left) are sketches of a fragmented segments: case a represents a segment consisting of one fragment; cases b and c show segments consisting of 2 (b) and 3 (c) fragments that are not connected. The resulting Euler characteristic values χ provided in the table reflect this fragmentation. The Euler characteristic χ is an **INVARIANT** that combines the numbers of vertices (V), edges (E), faces (F) and cells (C) in a single quantity, χ. **BETTI NUMBERS** are additional **INVARIANTS** that can be used to characterize the topology of an object at a high level of abstraction. The figure (left) shows a 3D solid object (3-dimensional manifold) consisting of one fragment, with one "hole," one "bubble" and one "tunnel." ...



Cases a, b and c show fragmentation of one segment into multiple fragments. a: segment has one fragment; b: segment has two fragments; c: segment has three fragments. Fragments are not connected.

case	V	E	F	χ
a	8	10	3	1
b	10	11	3	2
c	12	12	3	3

Associated Euler characteristics.

3D solid (3-manifold) with one "hole," one circular, closed "tunnel," and one "bubble" cavity.