■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: ... **Betti numbers β_i, $i = 0, 1, \ldots$, describe the topology of an object at a "finer level of detail" than the Euler characteristic** (being just a single number). Considering the case of a 3D/volumetric/3-manifold segment, the Betti numbers β_i, $i = 0 \ldots 3$, can be used to represent the number of fragments defining the segment; the numbers of "holes" going through the segments; the numbers of "bubbles" in the interior of the fragments; and the closed, circular "tunnels" in the interior of the fragments. The **Betti numbers also define the Euler characteristic value via an alternating sum:**

$$\chi = \beta_0 - \beta_1 + \beta_2 - \beta_3 .$$

In our application, analyzing the voxel representation of a possibly fragmented segment in 3D space, we would use the set of voxels to define the complex for which to compute the β_i values (voxels as cells, voxel faces, voxel edges and voxel corners).

For our purposes, Betti number computation is computationally too expensive.
÷ BH

• Meaning of some Betti numbers:

β_0: number of connected components/fragments

β_1: number of "circular holes," "tunnels"

β_2: number of 2-dimensional "void," "cavities"

• **The computation of Betti numbers is expensive for large voxel segment with multiple fragments.**

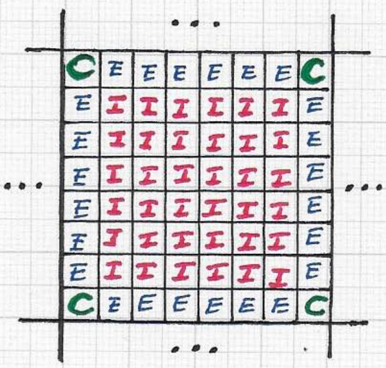
• Reference:

"The Topology of the Cosmic Web in Terms of Persistent Betti Numbers" by P. Pranav, H. Edelsbrunner et al., Monthly Notices of the Royal Astronomical Society 465(4), 2017.

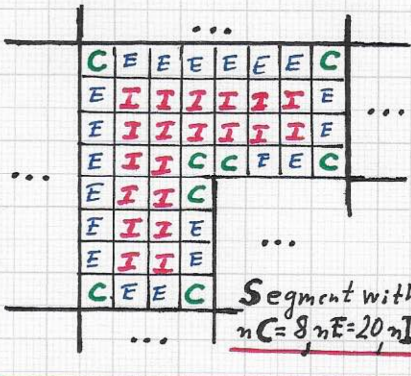
- Morse theory
- Persistence
- Filtrations
- Isosurfaces/contours
- Genus
- Euler characteristic
- Betti numbers
- Computation

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

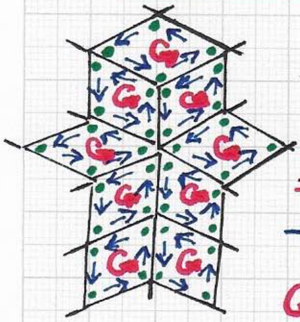
• Laplacian eigenfunctions:



Square texel with $p^2 = 8 \times 8$ pixels of types C (corner), E (edge) and I (interior). The texel is of type m or M. The numbers of pixels of the three pixel types are $nC = 4$, $nE = 4(p-2)$ and $nI = (p-2)^2$, respectively. Here: $nC=4$, $nE=24$, $nI=36$.



Segment with $nC=8$, $nE=20$, $nI=20$



Abstract sketch of more general complex of segments:
• corner
→ oriented edge
G oriented interior

The distribution of triples (nC, nE, nI) characterizes the COMPLEX.

The Euler characteristic and Betti numbers could potentially be helpful for the analysis and classification of 3D (3-manifold) segments, "complexes," consisting of voxels. Unfortunately, computing these features is prohibitively expensive and cannot be employed when a large number of segments of high voxel resolutions must be classified in a very short amount of time.

Thus, we re-consider the discussion of "texel corners," "texel edges" and "texel interiors" (see pp. 4 – 8, from 11/9/21 – 11/12/21). The shown figure (left, top) reminds us of the three types of "topological primitives" of a square texel of resolution $p \times p$: The σ-tuple computation of the pixels makes it possible to determine pixels of type C (0-dimensional texel corner pixels); pixels of type E (1-dimensional texel edge/boundary pixels); and pixels of type I (2-dimensional texel interior pixels). (In the case of a 3D texel, one has to consider voxels of four types: corner [0-dim.]; edge [1-dim.]; face [2-dim.]; and cell [3-dim.]). This information could be helpful for classifying a segment. ...

Bernd Hamann

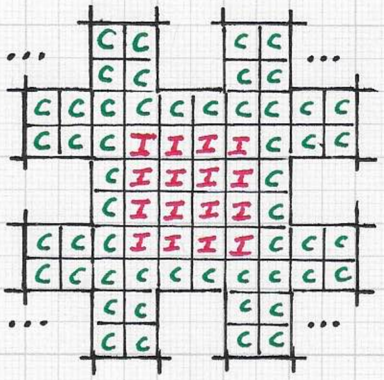
Stratovan

11/26/2021

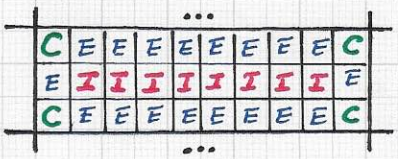
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: ... The non-square texels shown on the last page (left, images at bottom) and this page (left, images at top) are examples demonstrating that the values of nC, nE and nI (corner, edge and interior pixels), together with the number of pixels used to represent the texel (= nC+nE+nI), "encode" relevant characteristics of a texel: area of a texel; length of the boundary of a texel; degree of "oscillation" of the boundary of a texel (indicated by the number of pixels of type C); degree of "line" behavior of a texel (indicated by the number of pixels of type E relative to the number of pixels of type I) etc. Considering the ideal case of a square texel consisting of p^2 pixels (all having m or M as their mass value), pixels of type I quickly become the dominant type. The figures (left, bottom) show low-resolution texels of square resolution; the table provides numerical examples. ...

p	nC	nE	nI
1	1	0	0
2	4	0	0
3	4	4	1
4	4	8	4
6	4	16	16
8	4	24	36
16	4	56	196
32	4	120	900

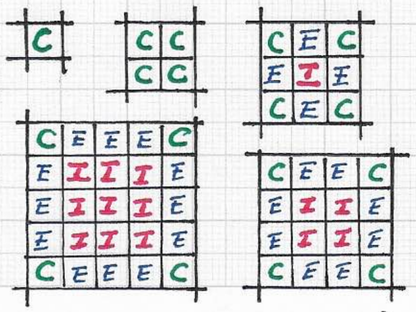


Segment with only two pixel types, C and I. Here, nC = 52, nI = 16, nE = 0. This example shows a "corner-dominant" segment.



Example of a segment of type "edge-dominant." Here, nC = 4, nE = 18, nI = 8.

The examples of these non-square texels show that the values of nC, nE and nI (their ratios) capture geometrical features.



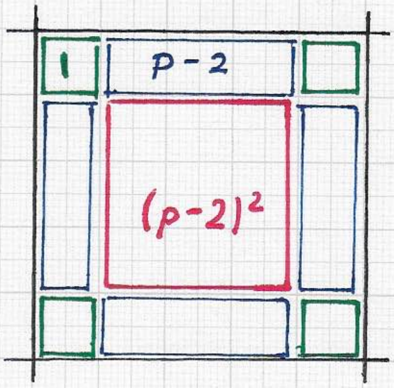
Texels of resolution p^2 ($p = 1, 2, 3, 4, 5$) and encoding of pixel types (C, E, I).

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

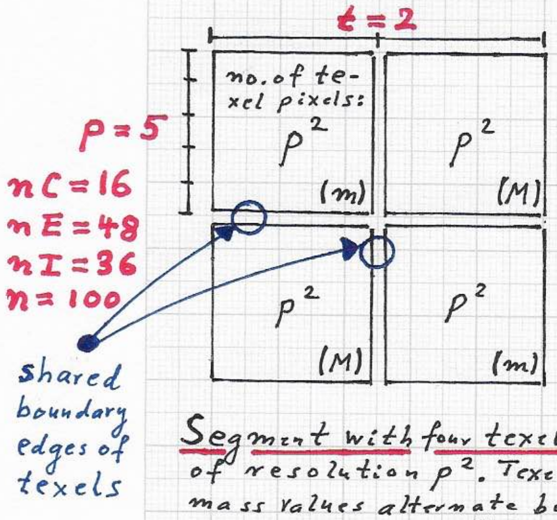
• Laplacian eigenfunctions: ... Considering the table provided on the last page and the figure shown here (left, top), $nC = 1$, all texels, implies that a segment has alternating mass values m and M at the pixel level and that all texels are of of resolution 1 pixel. If $nC = 4$ and $nE = nI = 0$, all texels, then a segment will have texels of resolution 2×2 pixels, all being of type C. For cases where $p > 2$, nC always has the constant value $nC = 4$; one can define a measure for the "granularity" of a segment by the ratio nE/nI:

$$nE/nI = 4(p-2)/(p-2)^2 = 4/(p-2).$$

This measure assigns relatively lower values of "granularity" to segments with texels of relatively higher resolution $p \times p$.



Segment texel with 4 corner pixels, $4(p-2)$ edge pixels and $(p-2)^2$ interior pixels. In the ideal checkerboard case, the segment consists of t^2 texels with these numbers of the three pixel types. The texel-associated mass values alternate between m and M.



Segment with four texels of resolution p^2. Texel mass values alternate between m and M. Note: The shared edges of neighbor texels are drawn twice; those shared edges can be viewed as one or two edges, depending on context.

• **EQUATIONS:** [total nos. of m-value, M-value pixels known]

$$n = nC + nE + nI$$

$$n = t^2 p^2 \quad \Rightarrow \quad t = \frac{1}{p}\sqrt{n}$$

$$nC = 4t^2 \quad \left(\Rightarrow \quad t = \frac{1}{2}\sqrt{nC}\right)$$

$$\left.\begin{array}{l} nE = 4t^2(p-2) \\ nI = t^2(p-2)^2 \end{array}\right\} \Rightarrow \quad p = 2 + 4\,nI/nE \quad !$$

Known: nC, nE, nI – numbers of corner, edge, interior pixels (total values, considering all n segment pixels)

unknown: t^2, p^2 – numbers of texels and pixels-per-texel

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions:

$t=3$ $\quad p=4$

C	E	E	C	C	E	E	C	C	E	E	C
E	I	I	E	E	I	I	E	E	I	I	E
E	I	I	E	E	I	I	E	E	I	I	E
C	E	E	C	C	E	E	C	C	E	E	C
C	E	E	C	C	E	E	C	C	E	E	C
E	I	I	E	E	I	I	E	E	I	I	E
E	I	I	E	E	I	I	E	E	I	I	E
C	E	E	C	C	E	E	C	C	E	E	C
C	E	E	C	C	E	E	C	C	E	E	C
E	I	I	E	E	I	I	E	E	I	I	E
E	I	I	E	E	I	I	E	E	I	I	E
C	E	E	C	C	E	E	C	C	E	E	C

Segment with $t \times t = 3 \times 3 = 9$ texels and $p \times p = 4 \times 4 = 16$ pixels per texel. The texels are of alternating m-type and M-type, i.e., the pixels in a specific texels all have mass value m or M. All pixels have already been classified as corner (C), edge (E) or interior (I) pixels.

Characterization of the segment is based on this input (data known after pixel classification):

f_i value of i^{th} pixel, $f_i \in \{m, M\}$

n total no. of pixels of segment

$M_T = \sum f_i$ total mass

$A = n$ total area

nC, nE, nI total numbers of pixels classified as C, E or I pixels

The segment shown (left) is a "checkerboard segment" consisting of $3 \times 3 = 9$ texels of "mass-type" m or M. Each texel has $4 \times 4 = 16$ pixels, and pixels in a texel have either all the value m or M. Geometrically, all pixels are $1 \times 1 = 1$ unit, square pixels. Each of the 144 pixels has been classified as a texel-corner (C), texel-edge (E) or texel-interior (I) pixel. It is now possible to characterize, analyze, the segment of 144 pixels with values m or M using the pixel classifications C, E and I at multiple levels of detail or abstraction — WITHOUT EXPLICIT KNOWLEDGE OF THE TEXELS. FROM THE "INPUT DATA" (left, bottom) ONE CAN DERIVE THE FOLLOWING:

$t = \frac{1}{2}\sqrt{nC}$ $\quad$ $(nC = 4t^2)$ "texel resolution"

$p = 2 + 4\,\frac{nI}{nE}$ $\quad$ $(nE = 4t^2(p-2)$ and $nI = t^2(p-2)^2)$

"pixel resolution per texel"

$t = \frac{1}{p}\sqrt{n}$ $\quad$ $(n = t^2 p^2)$

$T_m = (M t^2 - M_T / p^2) / (M - m)$

$T_M = t^2 - T_m$ $\quad$ $(M_T = m p^2 T_m + M p^2 T_M$
$\quad = m p^2 T_m + M p^2 (t^2 - T_m))$

T_m m-type texels

T_M M-type texels

$a = A / t^2$ area of a texel

...