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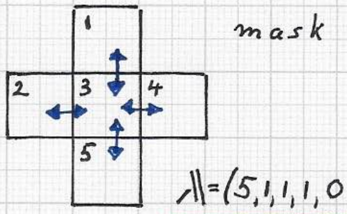
Stratovan

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions:



$\lambda = (5,1,1,1,0)$

$\mathbb{e}_1 = (1, 1, -4, 1, 1)^T$

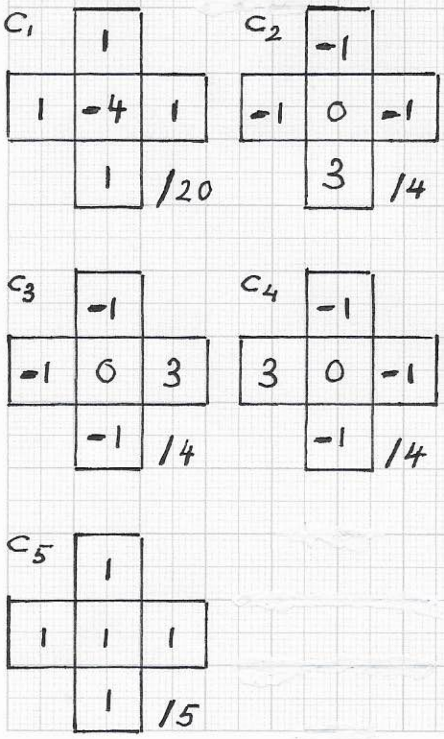
$\mathbb{e}_2 = (-1, 0, 0, 0, 1)^T$

$\mathbb{e}_3 = (-1, 0, 0, 1, 0)^T$

$\mathbb{e}_4 = (-1, 1, 0, 0, 0)^T$

$\mathbb{e}_5 = (1, 1, 1, 1, 1)^T$

$\Rightarrow \mathbb{f} = \sum_{i=1}^{5} c_i \mathbb{e}_i$



5-pixel mask and computation of coefficients c_i for eigenfunctions $\mathbb{e}_i$.

Generally, it is not necessary to normalize the eigenvectors/-functions associated with a convolution mask. Considering the last example based on a 5-pixel mask (see pp. 11-12, 12-1-21, 12-2-21), using eigenvectors/-functions $\mathbb{e}_i$, $i=1\ldots5$, for the expansion of a function $\mathbb{f}$ suffices; it is not necessary to normalize $\mathbb{e}_i$ to define $\mathbb{e}_i^n$. It can be computationally advantageous to avoid normalization: one can reduce the number of multiplications and/or numerical errors. Of course, the coefficients c_i of an expansion $\mathbb{f} = \sum_{i=1}^{5} c_i \mathbb{b}_i$ depend on the basis vectors/functions $\mathbb{b}_i$, and using $\{\mathbb{e}_i\}$ or $\{\mathbb{e}_i^n\}$ as bases leads to different c_i-values. When using the eigenfunctions $\mathbb{e}_i$ shown in the figure (left) one obtains $\mathbb{C}$ as follows:

$$\mathbb{C} = \left[\langle \mathbb{e}_i, \mathbb{e}_j \rangle \right]^{-1} \left[\langle \mathbb{f}, \mathbb{e}_i \rangle \right]$$

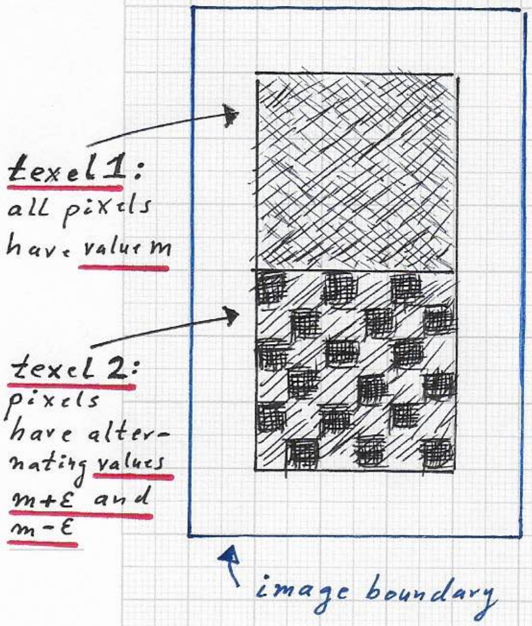
$$= \begin{bmatrix} 20 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}^{-1} \begin{bmatrix} m_1 + m_2 - 4m_3 + m_4 + m_5 \\ -m_1 + m_5 \\ -m_1 + m_4 \\ -m_1 + m_2 \\ m_1 + m_2 + m_3 + m_4 + m_5 \end{bmatrix}$$

$= \ldots =$ ...

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: ...

$$= \begin{bmatrix} (m_1+m_2-4m_3+m_4+m_5)/20 \\ (-m_1-m_2-m_4+3m_5)/4 \\ (-m_1-m_2+3m_4-m_5)/4 \\ (-m_1+3m_2-m_4-m_5)/4 \\ (m_1+m_2+m_3+m_4+m_5)/5 \end{bmatrix}.$$

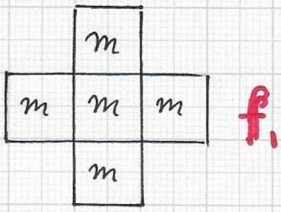


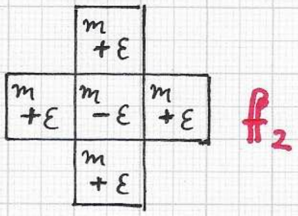
Example of a segment with one fragment consisting of two texels.

Note. It is interesting and important to observe that, in this example, there is no need to perform square root computations when calculating the coefficients c_i of the function $f = \sum_{i=1}^{5} c_i\, e_i$.

A simple example of a one-fragment segment is shown in the figure (left). We are interested in the e-tuples generated by the 5-pixel convolution mask based on the eigenfunctions e_i. (For simplicity, we only discuss the cases when all 5 mask pixels are entirely in the interior of texel 1 or texel 2.)

We calculate the coefficients c_i when the convolution mask is in the interior of texel 1 and texel 2.





5-pixel mask covering an interior region in texel 1 (top) and in texel 2 (bottom).

Texel 1:

$c_1 = (m+m-4m+m+m)/20 = 0$

$c_2 = (-m-m-m+3m)/4 = 0$

$c_3 = (-m-m+3m-m)/4 = 0$

$c_4 = (-m+3m-m-m)/4 = 0$

$c_5 = (m+m+m+m+m)/5 = m$ ← average value of texel

Texel 2:

$c_1 = (4(m+\varepsilon)-4(m-\varepsilon))/20 = \frac{2}{5}\varepsilon$

$c_2 = c_3 = c_4 = (3(m+\varepsilon)-3(m+\varepsilon))/4 = 0$

$c_5 = (4(m+\varepsilon)+(m-\varepsilon))/5 = m + \frac{3}{5}\varepsilon$

$f_1 = m\, e_5$

$f_2 = \frac{2}{5}\varepsilon\, e_1 + (m + \frac{3}{5}\varepsilon)\, e_5$

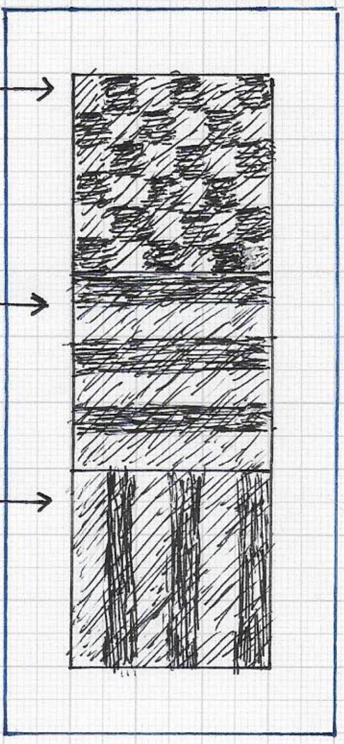
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: We provide an additional example. This example considers a segment with one fragment of three texels: Texel 3 is dual to texel 2 (black and white pixels exchanged); texels 4 and 5 exhibit "stripe textures." The coefficients c_i are:

texel 3: all pixels have values $m+\varepsilon$ or $m-\varepsilon$

texel 4: all pixels have values $m+\varepsilon$ or $m-\varepsilon$

texel 5: "texel-4 texture rotated by 90°"



Texel 3:

$$c_1 = (4m-4\varepsilon-4m-4\varepsilon)/20 = -\tfrac{2}{5}\varepsilon$$
$$c_2 = c_3 = c_4 = (3(m-\varepsilon)-3(m-\varepsilon))/4 = 0$$
$$c_5 = (5m-3\varepsilon)/5 = m-\tfrac{3}{5}\varepsilon$$

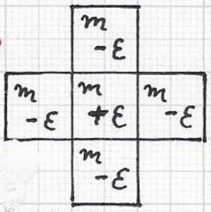
Texel 4:

$$c_1 = (4m+2\varepsilon-4m-4\varepsilon)/20 = -\varepsilon/10$$
$$c_2 = (3m-3m-2\varepsilon)/4 = -\varepsilon/2$$
$$c_3 = (3m+3\varepsilon-3m-\varepsilon)/4 = \varepsilon/2$$
$$c_4 = (3m+3\varepsilon-3m-\varepsilon)/4 = \varepsilon/2$$
$$c_5 = (5m+3\varepsilon)/5 = m+{}^3/_5\,\varepsilon$$

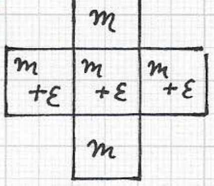
Texel 5:

$$c_1 = (4m+2\varepsilon-4m-4\varepsilon)/20 = -\varepsilon/10$$
$$c_2 = (3m+3\varepsilon-3m-\varepsilon)/4 = \varepsilon/2$$
$$c_3 = (3m-3m-2\varepsilon)/4 = -\varepsilon/2$$
$$c_4 = (3m-3m-2\varepsilon)/4 = -\varepsilon/2$$
$$c_5 = (5m+3\varepsilon)/5 = m+{}^3/_5\,\varepsilon$$

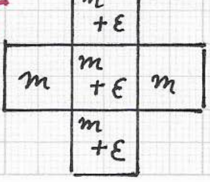
ff_3



ff_4



ff_5



Example of a segment with one fragment consisting of three texels.

The 5-pixel mask when applied to the interior of the texels returns the functions ff_3, ff_4 or ff_5.

Thus, the "texel function expansions" are:

$$ff_3 = -\tfrac{2}{5}\varepsilon\, e_1 + \left(m-\tfrac{3}{5}\varepsilon\right) e_5$$

$$ff_4 = -\tfrac{1}{10}\varepsilon\, e_1 + (-e_2+e_3+e_4)\,\varepsilon/2 + \left(m+\tfrac{3}{5}\varepsilon\right) e_5$$

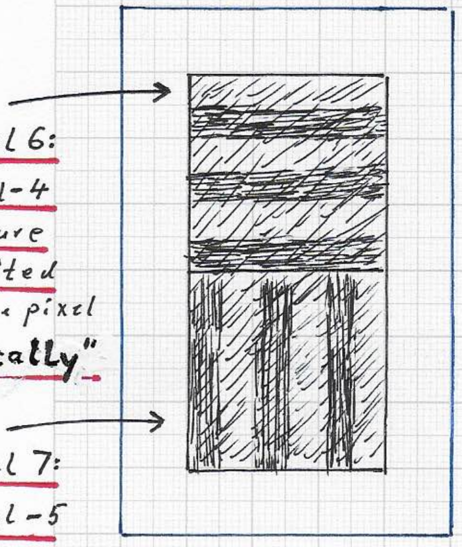
$$ff_5 = -\tfrac{1}{10}\varepsilon\, e_1 + \tfrac{1}{2}\varepsilon\,(e_2-e_3-e_4) + \left(m+\tfrac{3}{5}\varepsilon\right) e_5$$

Note. Texels 2 and 3 have the same alternating black-white checkerboard texture. **Functions ff_2 and ff_3** have coefficients c_1 and c_5 that express the "dual nature": $c_1 = +/- \tfrac{2}{5}\varepsilon$ and $c_5 = m +/- \tfrac{3}{5}\varepsilon$.

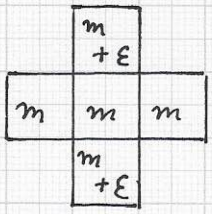
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: Concerning texels 4 and 5, we have provided convolution mask responses, coefficient ⓒ-tuples, only for the case when the mask's center pixel has the value $(m+\varepsilon)$. Here, we also present the "dual case," i.e., the case when the mask's center pixel has the value m. Qualitatively, texel 4 and texel 6 (texel 5 and texel 7) are the same; the stripe textures of texels 6 and 7 are only shifted. The resulting coefficient values are:

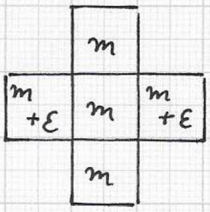
texel 6: "texel-4 texture shifted by one pixel **vertically**"



texel 7: "texel-5 texture shifted by one pixel **horizontally**"



ff₆



ff₇

Texel 6:

$$c_1 = (4m+2\varepsilon-4m)/20 = \varepsilon/10$$
$$c_2 = (3m+3\varepsilon-3m-\varepsilon)/4 = \varepsilon/2$$
$$c_3 = (3m-3m-2\varepsilon)/4 = -\varepsilon/2$$
$$c_4 = (3m-3m-2\varepsilon)/4 = -\varepsilon/2$$
$$c_5 = (5m+2\varepsilon)/5 = m+2/5\,\varepsilon$$

Texel 7:

$$c_1 = (4m+2\varepsilon-4m)/20 = \varepsilon/10$$
$$c_2 = (3m-3m-2\varepsilon)/4 = -\varepsilon/2$$
$$c_3 = (3m+3\varepsilon-3m-\varepsilon)/4 = \varepsilon/2$$
$$c_4 = (3m+3\varepsilon-3m-\varepsilon)/4 = \varepsilon/2$$
$$c_5 = (5m+2\varepsilon)/5 = m+2/5\,\varepsilon$$

Thus, the expansions for ff₆ and ff₇ are:

$$ff_6 = \tfrac{1}{10}\varepsilon\, ⓒ_1 + \tfrac{1}{2}\varepsilon\,(ⓒ_2 - ⓒ_3 - ⓒ_4) + (m+\tfrac{2}{5}\varepsilon)\, ⓒ_5$$

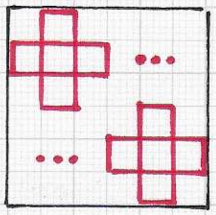
$$ff_7 = \tfrac{1}{10}\varepsilon\, ⓒ_1 + \tfrac{1}{2}\varepsilon\,(-ⓒ_2 + ⓒ_3 + ⓒ_4) + (m+\tfrac{2}{5}\varepsilon)\, ⓒ_5$$

Example of shifted versions of texel textures. Texels 6 and 7 are qualitatively identical to texels 4 and 5, except that their stripe textures are shifted by one pixel.

Functions ff₆ and ff₇ are the "dual" functions of ff₄ and ff₅, where mask mass values are also shifted by one pixel.

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions:



Texel used in examples: Texel resolution is 6×6 pixels; 16 $(=4\times4)$ 5-pixel masks are entirely inside the texel.

		$\mathbb{C}$-tuple	#
CONSTANT	$\mathbb{f}_1$	$0,0,0,0,m$	16
CHECKER-BOARD	$\mathbb{f}_2$	$\frac{2}{5}\varepsilon, 0, 0, 0, m+\frac{3}{5}\varepsilon$	8
	$\mathbb{f}_3$	$-\frac{2}{5}\varepsilon, 0, 0, 0, m-\frac{3}{5}\varepsilon$	8
STRIPES	$\mathbb{f}_4$	$-\frac{1}{10}\varepsilon, -\frac{1}{2}\varepsilon, \frac{1}{2}\varepsilon, \frac{1}{2}\varepsilon, m+\frac{3}{5}\varepsilon$	8
	$\mathbb{f}_5$	$-\frac{1}{10}\varepsilon, \frac{1}{2}\varepsilon, -\frac{1}{2}\varepsilon, -\frac{1}{2}\varepsilon, m+\frac{3}{5}\varepsilon$	8
	$\mathbb{f}_6$	$\frac{1}{10}\varepsilon, \frac{1}{2}\varepsilon, -\frac{1}{2}\varepsilon, -\frac{1}{2}\varepsilon, m+\frac{2}{5}\varepsilon$	8
	$\mathbb{f}_7$	$\frac{1}{10}\varepsilon, -\frac{1}{2}\varepsilon, \frac{1}{2}\varepsilon, \frac{1}{2}\varepsilon, m+\frac{2}{5}\varepsilon$	8

The "$\mathbb{C}$-tuple responses" of 5-pixel mask when applied to the 4×4 5-pixel region lying entirely inside the 6×6 texel. $\mathbb{f}_1$'s $\mathbb{C}$-tuple is returned 16 times (=#) for a CONSTANT function; The $\mathbb{C}$-tuples for $\mathbb{f}_2$ and $\mathbb{f}_3$ are returned 8 times each for the CHECKERBOARD function; and the $\mathbb{C}$-tuples for $\mathbb{f}_4$ and $\mathbb{f}_6$ ($\mathbb{f}_5$ and $\mathbb{f}_7$) are returned 8 times each for the horizontal (vertical) STRIPE function.

It is relevant to consider the "meaning and interpretation" of the function expansions obtained for $\mathbb{f}_1, \dots, \mathbb{f}_7$. All texels used in the examples for the computation of these expansions are based on texels of pixel resolution 6×6. As a consequence 4×4 5-pixel masks have pixels that are all inside the 6×6 texel. Of interest are the "$\mathbb{C}$-tuple responses" for the three types of texel textures used: constant-type, checkerboard-type and stripe-type. As a reminder, $\mathbb{c}_1$ (and c_1) reflects highest-frequency information; $\mathbb{c}_2, \mathbb{c}_3$ and $\mathbb{c}_4$ (c_2, c_3 and c_4) reflect medium-frequency information; and $\mathbb{c}_5$ (and c_5) reflects lowest-frequency information present in a "5-pixel signal."

• Meaning and Interpretation:

$\mathbb{f}_1$: The spectral behavior of the constant function is captured completely with eigenfunction $\mathbb{c}_5$, which is simply multiplied by $c_5 = m$.

$\mathbb{f}_2, \mathbb{f}_3$: The average of all coefficient values for c_5 reflects the average value of mass: $\frac{1}{16}\left(8\left(m+\frac{3}{5}\varepsilon\right)+8\left(m-\frac{3}{5}\varepsilon\right)\right)=m$.

...