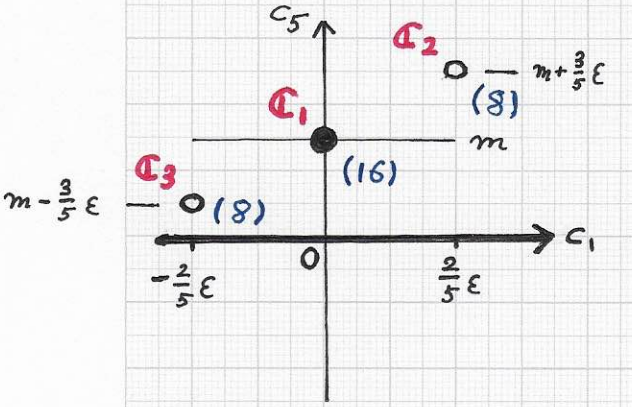
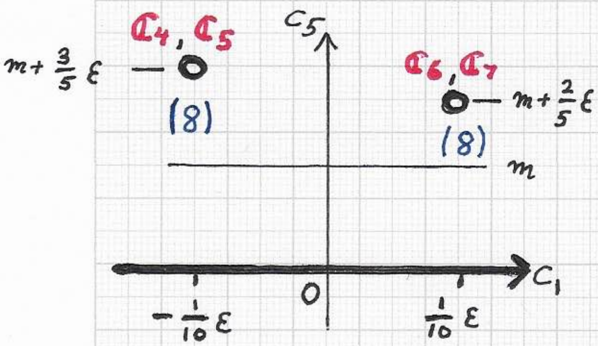
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions:...



Sketch of $\mathbb{C}$-tuples $\mathbb{C}_1$, $\mathbb{C}_2$ and $\mathbb{C}_3$ for functions $\mathbb{f}_1$, $\mathbb{f}_2$ and $\mathbb{f}_3$, respectively.

The $\mathbb{C}$-tuples are projected into the (c_1, c_5)-plane. The numbers in brackets "(·)" indicate how often the specific $\mathbb{C}$-tuple value is returned by the mask for 16 mask calculations.



Sketch of $\mathbb{C}$-tuples $\mathbb{C}_4$, $\mathbb{C}_5$, $\mathbb{C}_6$ and $\mathbb{C}_7$ for functions $\mathbb{f}_4$, $\mathbb{f}_5$, $\mathbb{f}_6$ and $\mathbb{f}_7$, respectively. The $\mathbb{C}$-tuples are projected into the (c_1, c_5)-plane. The horizontal stripes (vertical stripes) generate 8 times the tuple $\mathbb{C}_4$ ($\mathbb{C}_5$) and 8 times the tuple $\mathbb{C}_6$ ($\mathbb{C}_7$).

All values generated for c_2, c_3 and c_4 are 0. Using an "informal explanation," this is a consequence of the fact that eigenfunctions $\mathbb{e}_2$, $\mathbb{e}_3$ and $\mathbb{e}_4$ are basis functions to represent linear behavior over the 5-pixel mask region ("up-down" linear ramp behavior). The alternating checkerboard function does not exhibit such linear ramp behavior over the mask, and thus $c_2 = c_3 = c_4 = 0$ for all 16 pixels.

Eigenfunction $\mathbb{e}_1$ (and c_1) is used to represent "highest-frequency oscillation behavior." This type of behavior is present in the two functions ($\mathbb{f}_2$, $\mathbb{f}_3$), where $+\varepsilon$ or $-\varepsilon$ is added to the average value m. Thus, c_1 has the value $+\frac{2}{5}\varepsilon$ or $-\frac{2}{5}\varepsilon$.

$\mathbb{f}_4$, $\mathbb{f}_5$, $\mathbb{f}_6$, $\mathbb{f}_7$: Again, the average c_5-value represents the average mass value:

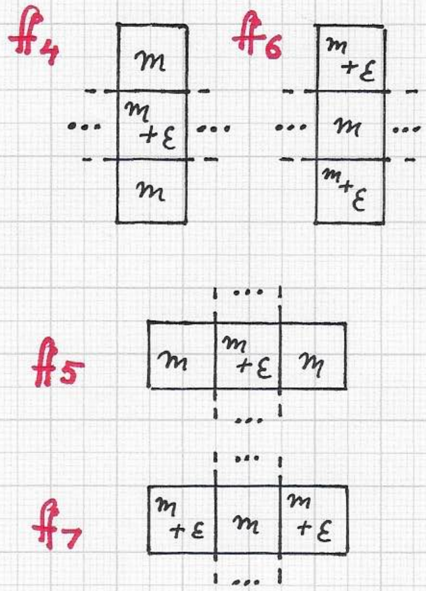
$$\frac{1}{16}\left(8\left(m+\frac{3}{5}\varepsilon\right)+8\left(m+\frac{2}{5}\varepsilon\right)\right) = m+\frac{1}{2}\varepsilon,$$

which results for the horizontal-stripes case ($\mathbb{f}_4$, $\mathbb{f}_6$) and the vertical-stripes case ($\mathbb{f}_5$, $\mathbb{f}_7$).

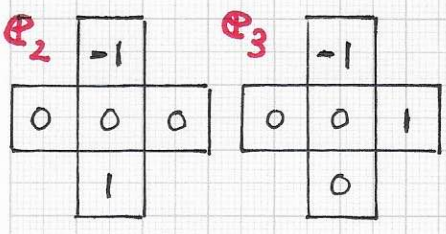
...

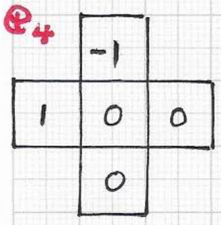
OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: ... The values of c_1-coefficients relate to highest-frequency information captured by the 5-pixel mask; the associated eigenfunction $e_1 = (1,1,-4,1,1)^T$ is the basis function needed to represent the "non-zero second derivative oscillation behavior" reflected in the 5 mass values in the 5-pixel mask. The figure (left) shows the 3 mass values that generate the resulting c_1 values of $-\frac{1}{10}\varepsilon$ and $\frac{1}{10}\varepsilon$ for the horizontal and vertical stripe textures. The mass value sequence $m, (m+\varepsilon), m$ leads to the result $-\frac{1}{10}\varepsilon$, and the sequence $(m+\varepsilon), m, (m+\varepsilon)$ leads to the result $\frac{1}{10}\varepsilon$. Eigenfunctions e_2, e_3 and e_4 are associated with the triple eigenvalue $1 = \lambda_2 = \lambda_3 = \lambda_4$. Their linear combination in a function expansion is necessary to represent "middle-frequency behavior," related to linear ramp behavior in this case, which is represented in the mass value sequences $m, (m+\varepsilon)$ and $(m+\varepsilon), m$. The figure (left, bottom) shows the eigenfunctions e_2, e_3 and e_4. ...



Mask's recorded mass values for horizontal stripe texture (f4, f6) and vertical stripe texture (f5, f7), for the mask's "column of 3 pixels" (top) and "row of 3 pixels" (bottom).





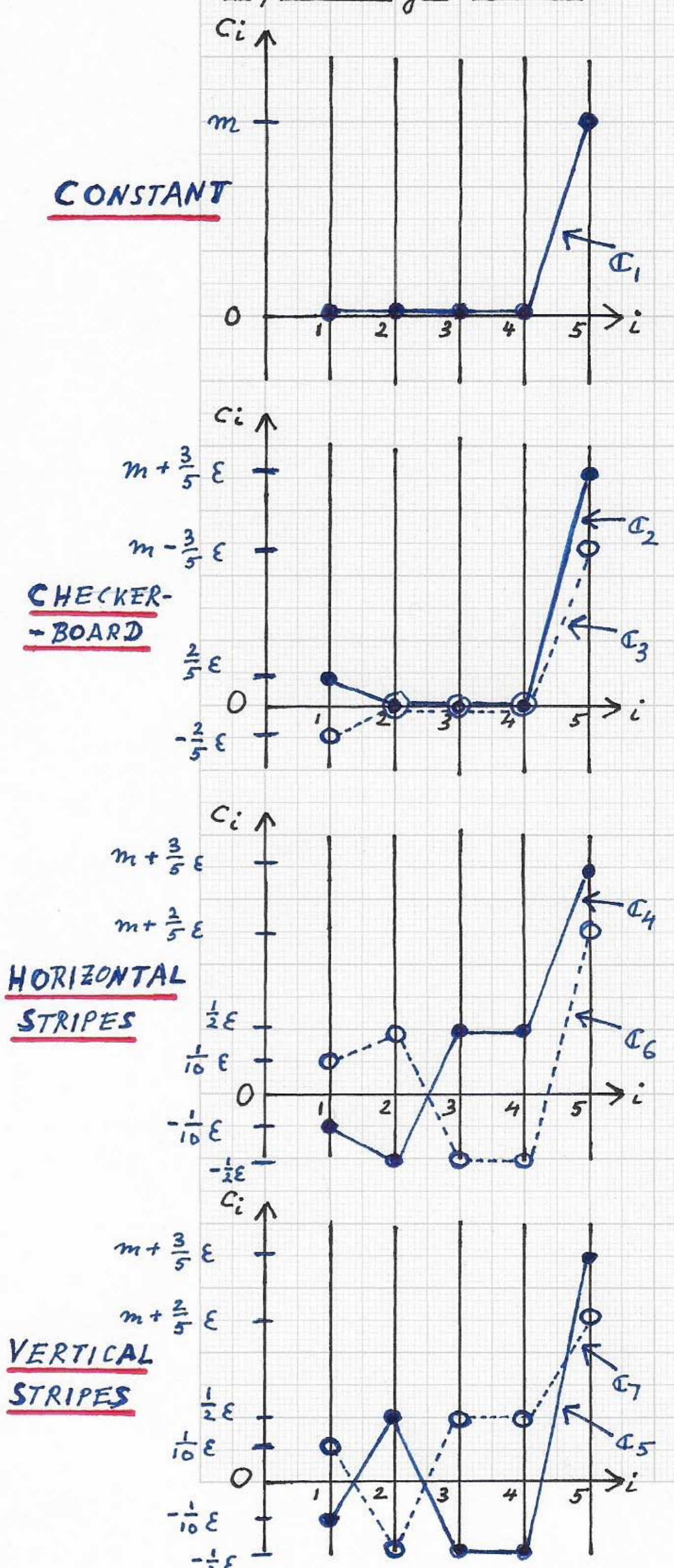
"Middle-frequency eigenfunctions exhibiting linear ramp behavior" via the number pair −1, 1.

Bernd Hamann

Stratovan

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions:



Parallel coordinates visualization of $\mathbb{C}$-tuples

The $\mathbb{C}$-tuples of functions $f_1, \ldots, f_7$ have associated values $\mathbb{C}_i = \mathbb{C}(f_i) = (c_1^i, c_2^i, c_3^i, c_4^i, c_5^i)$. For example, one can use "parallel coordinates" as visualization method to understand the 5-dimensional nature of and relationships between $\mathbb{C}_i$-tuples. The figures shown here provide the parallel coordinates visualizations of $\mathbb{C}_1, \ldots, \mathbb{C}_7$. (Note: The values of m and ε are positive, $m, \varepsilon > 0$, and $m \gg \varepsilon$.)

Since $\lambda_5 = 0$ ($w_5 = 0$), eigenfunction e_5 and its associated coefficient c_5 relate to "average function value" – i.e., lowest-frequency ("zero-frequency") information; this fact is reflected in the c_5-values of the functions shown in the "parallel coordinates" plots. Further, the checkerboard texture has mass values that alternate from pixel to pixel, and the horizontal (vertical) stripe texture has mass values that alternate from row to row (column to column). Thus, when ignoring coefficient c_5, $\mathbb{C}_3 = -\mathbb{C}_2$, $\mathbb{C}_6 = -\mathbb{C}_4$ and $\mathbb{C}_7 = -\mathbb{C}_5$.

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions:

• Function expressed in eigenfunctions:

$$\mathbb{f} = \sum_{i=1}^{5} c_i \mathbb{e}_i$$

LOW-FREQUENCY part:

$$\mathbb{f}^{LOW} = c_5 \mathbb{e}_5$$

MEDIUM-FREQUENCY part:

$$\mathbb{f}^{MED} = \sum_{i=2}^{4} c_i \mathbb{e}_i$$

HIGH-FREQUENCY part:

$$\mathbb{f}^{HIGH} = c_1 \mathbb{e}_1$$

⇒ The representation of a function $\mathbb{f}$ in the eigenbasis $\{\mathbb{e}_i\}$ makes it possible to analyze $\mathbb{f}$ in a multi-scale manner.

• $\lambda_5, \mathbb{e}_5, c_5 \Rightarrow \mathbb{f}^{LOW}$

• $\lambda_2, \lambda_3, \lambda_4, \mathbb{e}_2, \mathbb{e}_3, \mathbb{e}_4, c_2, c_3, c_4 \Rightarrow \mathbb{f}^{MED}$

• $\lambda_1, \mathbb{e}_1, c_1 \Rightarrow \mathbb{f}^{HIGH}$

[Since $\lambda_2 = \lambda_3 = \lambda_4 = 1$, $\mathbb{f}^{MED}$ is based on three eigenfunctions.]

By construction, the eigenfunctions $\mathbb{e}_i$ and their calculated coefficient $\mathbb{C}$-tuples support a hierarchical, multi-scale, level-of-detail reconstruction/approximation of the "5-pixel function" seen by the convolution mask. We discuss some aspects of this multi-scale nature of the eigenfunction-based representation as it could potentially be considered for two purposes: (i) performing image data analysis in a scale-specific/frequency-band-specific fashion or (ii) ignoring certain high-frequency eigenfunctions that relate to known noise in an image. We consider the constant texture function, the checkerboard texture function and the horizontal stripe texture function.

CONSTANT TEXTURE $\mathbb{f}_1$. This function leads to only one relevant coefficient, i.e., c_5. This function can be reconstructed without error as $\mathbb{f}_1 = \mathbb{f}^{LOW} = c_5 \mathbb{e}_5 = m(1,1,1,1,1)^T$.

CHECKERBOARD TEXTURE $\mathbb{f}_2$. The coefficient $\mathbb{C}$-tuple for this function is $\mathbb{C} = (c_1, c_2, c_3, c_4, c_5) = (\frac{2}{5}\varepsilon, 0, 0, 0, m + \frac{3}{5}\varepsilon)$.

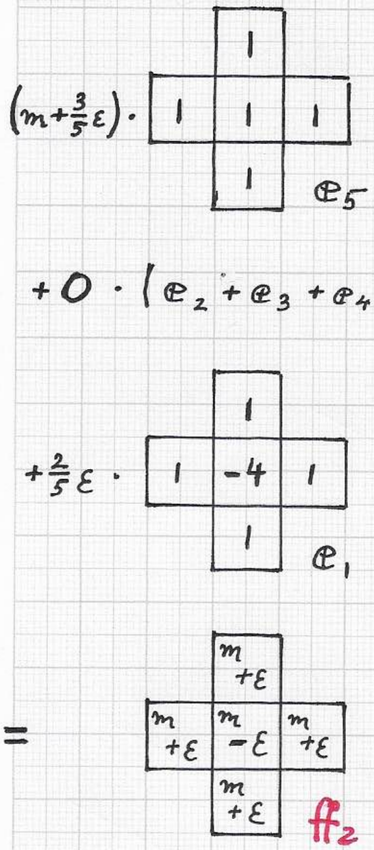
...

OBJECT AND MATERIAL EIGENFUNCTIONS — Cont'd.

• Laplacian eigenfunctions: ... For the checkerboard texture function $\mathbb{f}_2$ one obtains:

$$\mathbb{f}^{LOW} = \left(m+\tfrac{3}{5}\varepsilon\right)(1,1,1,1,1)^T$$
$$\mathbb{f}^{MED} = 0\,\mathbb{e}_2 + 0\,\mathbb{e}_3 + 0\,\mathbb{e}_4 = \mathbb{0}$$
$$\mathbb{f}^{HIGH} = \tfrac{2}{5}\varepsilon\,(1,1,-4,1,1)^T$$

Of interest are the errors that result when approximating a function $\mathbb{f}$ by $\mathbb{f}^{LOW}$, by $\mathbb{f}^{LOW}+\mathbb{f}^{MED}$ and by $\mathbb{f}^{LOW}+\mathbb{f}^{MED}+\mathbb{f}^{HIGH}$. For example, one can use the root-mean-square error and call the resulting RMS errors E^{L}, $E^{L,M}$ and $E^{L,M,H}$. Obviously, $E^{L,M,H}=0$ since the approximation is lossless when using all eigenfunctions.



Decomposition of $\mathbb{f}_2$ into its three distinct frequency components.

The sequence of RMS approximation errors is $E^{L}=\frac{4}{5}\varepsilon$, $E^{L,M}=\frac{4}{5}\varepsilon$ and $E^{L,M,H}=0$.

For $\mathbb{f}_2$ one obtains the differences:

$$\mathbb{f}_2-\mathbb{f}^{LOW} = (m+\varepsilon, m+\varepsilon, m-\varepsilon, m+\varepsilon, m+\varepsilon)^T$$
$$-\left(m+\tfrac{3}{5}\varepsilon, m+\tfrac{3}{5}\varepsilon, m+\tfrac{3}{5}\varepsilon, m+\tfrac{3}{5}\varepsilon, m+\tfrac{3}{5}\varepsilon\right)^T$$
$$=\left(\tfrac{2}{5}\varepsilon, \tfrac{2}{5}\varepsilon, -\tfrac{8}{5}\varepsilon, \tfrac{2}{5}\varepsilon, \tfrac{2}{5}\varepsilon\right)^T.$$

$$\Rightarrow E^{L} = \left(\tfrac{1}{5}\left(\tfrac{4}{25}\varepsilon^2\cdot 4 + \tfrac{64}{25}\varepsilon^2\right)\right)^{1/2} = \left(\tfrac{1}{5}\cdot\tfrac{80}{25}\varepsilon^2\right)^{1/2} = \tfrac{4}{5}\varepsilon$$

$$\mathbb{f}_2 - \left(\mathbb{f}^{LOW}+\mathbb{f}^{MED}\right) = \ldots = \mathbb{f}_2 - \mathbb{f}^{LOW}$$

$$\Rightarrow E^{L,M} = E^{L} = \tfrac{4}{5}\varepsilon$$

$$\mathbb{f}_2 - \left(\mathbb{f}^{LOW}+\mathbb{f}^{MED}+\mathbb{f}^{HIGH}\right) = \ldots = (0,0,0,0,0)^T = \mathbb{0}$$

$$\Rightarrow E^{L,M,H} = 0$$

...