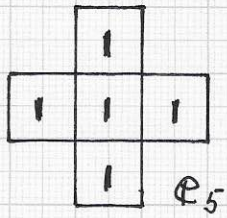
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

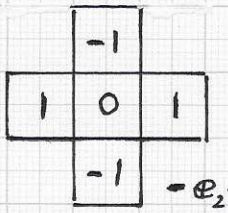
•Laplacian eigenfunctions: ... HORIZONTAL STRIPE TEXTURE f_4. The coefficient $\mathbb{C}$-tuple for this function is $\mathbb{C} = (-\frac{1}{10}\varepsilon, -\frac{1}{2}\varepsilon, \frac{1}{2}\varepsilon, \frac{1}{2}\varepsilon, m+\frac{3}{5}\varepsilon)$. For this function one obtains:

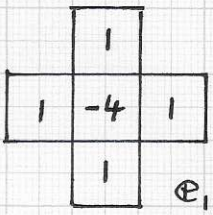
$$\underline{ff^{LOW}} = (m+\tfrac{3}{5}\varepsilon)(1,1,1,1,1)^T$$

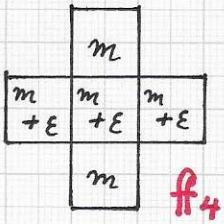
$$\underline{ff^{MED}} = -\tfrac{1}{2}\varepsilon(-1,0,0,0,1)^T + \tfrac{1}{2}\varepsilon(-1,0,0,1,0)^T + \tfrac{1}{2}\varepsilon(-1,1,0,0,0)^T = \varepsilon(-\tfrac{1}{2},\tfrac{1}{2},0,\tfrac{1}{2},-\tfrac{1}{2})^T$$

$$\underline{ff^{HIGH}} = -\tfrac{1}{10}\varepsilon(1,1,-4,1,1)^T$$

$(m+\frac{3}{5}\varepsilon)\cdot$  $\mathbb{C}_5$

$+\frac{1}{2}\varepsilon\cdot$  $-\mathbb{C}_2+\mathbb{C}_3+\mathbb{C}_4$

$-\frac{1}{10}\varepsilon\cdot$  $\mathbb{C}_1$

$=$  f_4

Again, we calculate RMS errors:

$$\underline{f_4 - ff^{LOW}} = (m, m+\varepsilon, m+\varepsilon, m+\varepsilon, m)^T - (m+\tfrac{3}{5}\varepsilon)(1,1,1,1,1)^T = (-\tfrac{3}{5}\varepsilon, \tfrac{2}{5}\varepsilon, \tfrac{2}{5}\varepsilon, \tfrac{2}{5}\varepsilon, -\tfrac{3}{5}\varepsilon)^T$$

$$\Rightarrow \underline{E^L} = \varepsilon\left(\left(2\cdot\tfrac{9}{25} + 3\cdot\tfrac{4}{25}\right)\cdot\tfrac{1}{5}\right)^{1/2} = \varepsilon\left(\tfrac{6}{25}\right)^{1/2} = \tfrac{1}{5}\varepsilon\sqrt{6}$$

$$\underline{f_4 - (ff^{LOW} + ff^{MED})} = \ldots = (-\tfrac{1}{10}\varepsilon, -\tfrac{1}{10}\varepsilon, \tfrac{2}{5}\varepsilon, -\tfrac{1}{10}\varepsilon, -\tfrac{1}{10}\varepsilon)^T$$

$$\Rightarrow \underline{E^{L,M}} = \varepsilon\left(\tfrac{1}{5}\left(4\cdot\tfrac{1}{100} + \tfrac{4}{25}\right)\right)^{1/2} = \varepsilon\left(\tfrac{1}{5}\cdot\tfrac{20}{100}\right)^{1/2} = \tfrac{1}{5}\varepsilon$$

$$\underline{f_4 - (ff^{LOW} + ff^{MED} + ff^{HIGH})} = \ldots = 0$$

$$\Rightarrow \underline{E^{L,M,H}} = 0$$

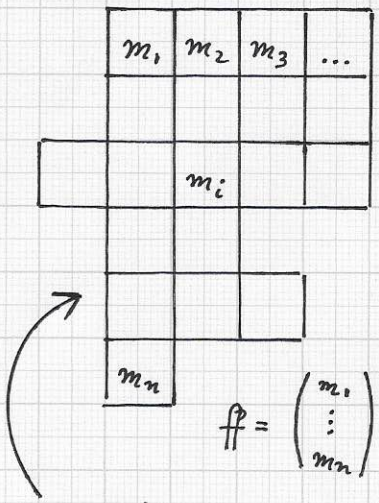
Decomposition of f_4 into its three distinct frequency components.

The sequence of RMS approximation errors is $E^L = \frac{\varepsilon}{5}\sqrt{6}$, $E^{L,M} = \varepsilon/5$ and $E^{L,M,H} = 0$.

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: The RMS errors are discussed here for the following main reason: Generally, the higher-frequency components of a function's expansion in a multi-frequency basis relate to noise and/or are insignificant for the function's analysis, characterization and classification. Further, for the purpose of computational and memory/storage efficiency it is highly desirable to use only a fraction, a part, of a function's expansion, i.e., the "minimally required terms" (="minimally required lower-frequency terms"), that suffices for a specific task. For example, a perfect, lossless expansion of a material's mass function might not be necessary for classification purposes; a small number of coefficients of lower-frequency eigenfunctions might be sufficient. **Thus**, when it is possible to define a priori how many low-frequency coefficients are needed of a **high-dimensional** $\mathbb{C}$-tuple, only these coefficients should be used as features.

Segment



grid/mesh

$\Rightarrow \lambda = (\lambda_1, \ldots, \lambda_n)$,

$\mathbb{e}_1, \ldots, \mathbb{e}_n$

$\Rightarrow \mathbb{m} = \sum_{i=1}^{n} c_i \mathbb{e}_i$,

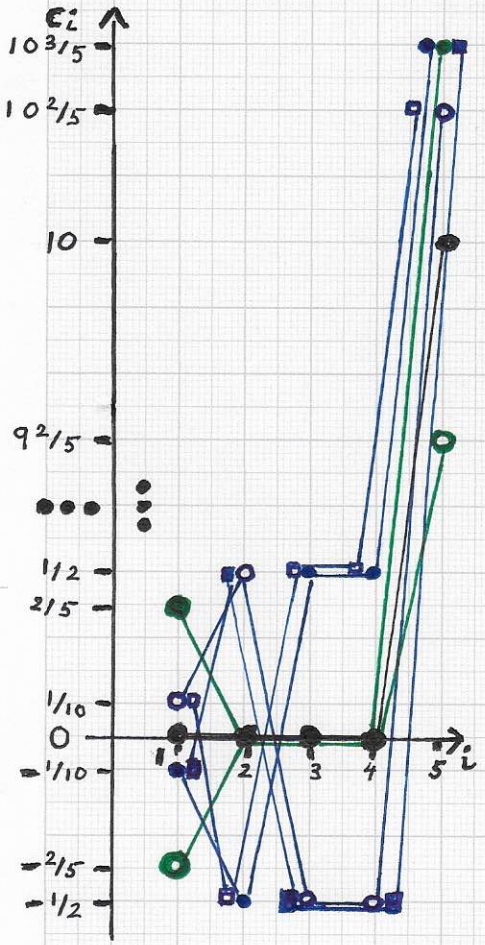
$\mathbb{m}_{app} = \sum_{I} c_i \mathbb{e}_i$,

where $I \subset \{1, 2, \ldots, n\}$

$\Rightarrow$ Use a SMALL subset of eigenfunctions $\mathbb{e}_i$ to approximate $\mathbb{m}$ by $\mathbb{m}_{app}$ for analysis and classification.

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: Visualizing multi-dimensional $\mathfrak{C}$-tuples with parallel coordinates plots can also be helpful for identifying very similar expansions of functions (identified with their respective coefficients in their expansions) or spotting function clusters. As an example, we plot the coefficient $\mathfrak{C}$-tuples for $ff_1, \ldots, ff_7$. The $\mathfrak{C}$-tuples for these functions are shown in parallel coordinates plots on p. 23 (12-11-2021). For the visualization shown here (left), we use the values $m = 10$ and $\varepsilon = 1$. The resulting parallel coordinates plots provide some "qualitative insights":



Parallel coordinates plots of $\mathfrak{C}$-tuples for $ff_1, \ldots, ff_7$. Colors used:

$\mathfrak{C}_1$
$\mathfrak{C}_2$
$\mathfrak{C}_3$
$\mathfrak{C}_4$
$\mathfrak{C}_5$
$\mathfrak{C}_6$
$\mathfrak{C}_7$

Plots can provide insight into frequency behavior.

→ The seven functions have approximately the same average value; the c_5-values lie between $9^2/5$ and $10^3/5$.

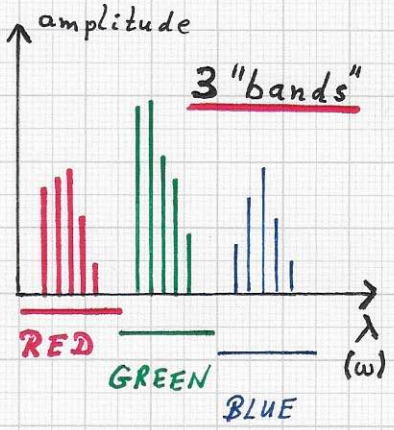
→ Functions ff_1, ff_2 and ff_3 all have medium-frequency coefficient values $c_2 = c_3 = c_4 = 0$; they do not contain medium-frequency information.

→ Only functions ff_4, ff_5, ff_6 and ff_7 contain medium-frequency information: $|c_2| = |c_3| = |c_4| = \frac{1}{2}$.

→ Functions ff_2 and ff_3 (checker board) exhibit the largest high-frequency behavior: $|c_1| = \frac{2}{5}$.

■ OBJECT AND MATERIAL EIGENFUNCTIONS - Cont'd.

• Laplacian eigenfunctions: When the number of dimensions of coefficient $\mathbb{C}$-tuples is "large" and/or the number of poly-lines plotted to represent many $\mathbb{C}$-tuples is "large", parallel coordinates plots will be hard to interpret. Besides such visual representations, there is a need to define and compute measures for difference or similarity of functions' frequency behavior via proper distance metrics for $\mathbb{C}$-tuples — for characterization, potential clustering and eventual classification. The expansion of the (mass) function defined over a pixel/voxel domain in terms of eigenfunctions directly supports the representation at multiple scales, i.e., multiple eigenfrequencies. Further, one can organize the function's eigenfrequency space into BANDS, i.e., into frequency intervals, that would make possible analysis and comparison of functions. In our (most recent) examples we merely used a 5-pixel mask for Local function expansions. In this case three different eigenvalues/-frequencies arise, and one can consider three frequencies ("frequency bands"): LOW, MEDIUM, HIGH. ...



Possibility to analyze spectral data based on "bands." The sketch illustrates the principle for a wavelength (frequency) domain called λ-domain (ω-domain). The domain is subdivided into three subdomains, called RED, GREEN and BLUE. The bars represent the amplitudes of specific wavelengths (frequencies).

Instead of using all the wavelengths (frequencies) present in a function's, a signal's, spectral expansion, it is possible and often desirable to compare two functions using only a specific band (or a few bands).

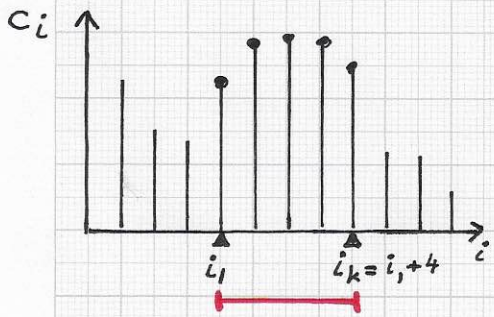
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

•Laplacian eigenfunctions: ... Since a function $\mathbf{ff} = (f_1, \ldots, f_n)^T$ is identified with its eigenfunction-based coefficient tuple $\mathbf{c} = (c_1, \ldots, c_n)$ of its multi-scale expansion $\mathbf{ff} = \sum_{i=1}^{n} c_i \mathbf{e}_i$ ($\mathbf{e}_i$ being the i^{th} eigenfunction), we need to define a band-specific way to compare $\mathbf{c}$-tuples of functions. Let functions $\mathbf{ff}_1$ and $\mathbf{ff}_2$ have associated $\mathbf{c}$-tuples $\mathbf{c}_1 = (c_1^1, \ldots, c_n^1)$ and $\mathbf{c}_2 = (c_1^2, \ldots, c_n^2)$. These two $\mathbf{c}$-tuples imply the difference tuple $d\mathbf{l} = (c_1^2 - c_1^1, \ldots, c_n^2 - c_n^1) = (d_1, \ldots, d_n)$. The difference tuple $d\mathbf{l}$ is a vector quantity, and we can define its length as

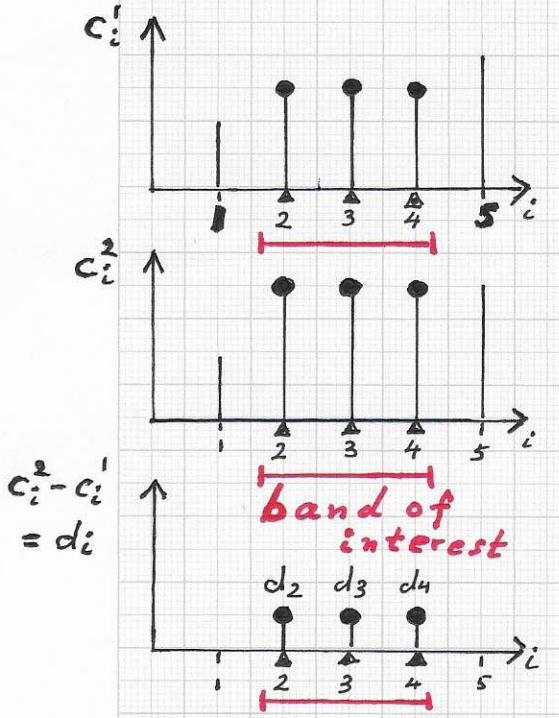
$$\| d\mathbf{l} \| = \sqrt{\langle d\mathbf{l}, d\mathbf{l} \rangle} = (d_1^2 + \ldots + d_n^2)^{1/2}.$$

This norm can be applied "selectively" by defining an **index set** $I \subseteq \{1, \ldots, n\}$, $I = \{i_1 < i_2 < \ldots < i_k\}$, $k \leq n$, which can also be written as $I = \{i_1, \ldots, i_k \mid i_1 < i_2 < \ldots < i_k\}$. Usually, a set I would consist of consecutive integer values, i.e., $I = \{i_1, i_1+1, i_1+2, \ldots, i_1+(k-1)\}$. A difference tuple can now be defined with respect to an index set I: $d\mathbf{l}_I = (c_{i_1}^2 - c_{i_1}^1, \ldots, c_{i_k}^2 - c_{i_k}^1) = (d_{i_1}, \ldots, d_{i_k})$; and $\| d\mathbf{l}_I \| = (d_{i_1}^2 + \ldots + d_{i_k}^2)^{1/2}$.

...



Example of an index set. The index set $I = \{i_1, i_1+1, \ldots, i_1+4\}$ is used to reference the 5 specific c_i-values of a band of interest.



Example based on 5-pixel mask. The "medium-frequency band" relates to coefficients c_2, c_3 and c_4. Defining the index set I as $I = \{2, 3, 4\}$, leads to $\| d\mathbf{l}_I \| = (d_2^2 + d_3^2 + d_4^2)^{1/2}$.