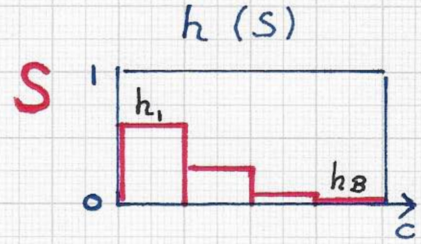
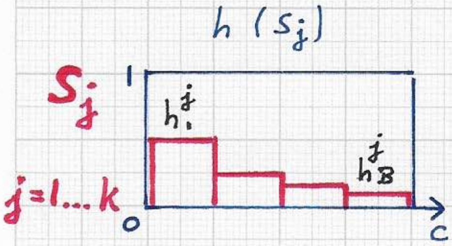
OBJECT AND MATERIAL EIGENFUNCTIONS — Cont'd.

• Laplacian eigenfunctions: ... In the following, we discuss the approximation/representation of the coefficient-value histogram of a segment to be analyzed and classified using the k stored sample coefficient-value histograms serving as examples of one class. Initially, the discussion will ignore the fact that this procedure ultimately must be performed for several scales involving histograms specific to the individual scales. It is assumed that histograms are discrete, consisting of B bins of equal width in the coefficient-value dimension (c). The goal is to express the given histogram $h(S)$ as a linear combination of the sample histograms $h(S_j)$, $j=1 \ldots k$:





B bins used for all histograms

Three cases must be considered when one wants to express the histogram $h(S)$ as a combination of stored histograms $h(S_j)$, with all histograms defined discretely via B bins. The three cases are:

- over-determined,
- determined,
- under-determined.

These cases correspond to these relationships between the numbers of bins (B) and samples (k):

- $B > k$,
- $B = k$,
- $B < k$.

$$
\begin{array}{ccccccc}
h_1^1 \alpha_1 & + & \ldots & + & h_1^k \alpha_k & = & h_1 \\
\vdots & & & & \vdots & & \vdots \\
h_B^1 \alpha_1 & + & \ldots & + & h_B^k \alpha_k & = & h_B
\end{array} ,
$$

$$
\begin{bmatrix} h_1^1 & \ldots & h_1^k \\ \vdots & & \vdots \\ h_B^1 & \ldots & h_B^k \end{bmatrix}
\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_k \end{bmatrix}
=
\begin{bmatrix} h_1 \\ \vdots \\ h_B \end{bmatrix} ,
$$

$$\mathbb{H} \quad \boldsymbol{\alpha} = \mathbb{h} \; . \quad !$$

...

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: ... Using vector notation, the goal is to represent $h(S)$ in the form

$$\mathbb{h} = \sum_{j=1}^{k} \alpha_j \mathbb{h}_j \quad .$$

The given histogram $h(S_j)$ is identified by the vector $(h_1^j, \ldots, h_B^j)^T$, and the unknown coefficients α_j in this expansion define the tuple $\alpha = (\alpha_1, \ldots, \alpha_k)$. IF $h(S)$ WERE TO MATCH EXACTLY ONE OF THE SAMPLE HISTOGRAMS $h(S_j)$ PERFECTLY, ONE'S GOAL WOULD BE TO OBTAIN THE TUPLE

$$\alpha = (0, \ldots, 0, \underset{\uparrow\, j^{th} \text{ component}}{1}, 0, \ldots, 0)$$

WHEN SOLVING THE SYSTEM OF LINEAR EQUATIONS FOR α.

Due to a variety of reasons, the "perfect-match case" generally cannot occur in a real-world scenario. Thus, when computing an α-tuple for the purpose of classification one must determine whether the α-tuple and/or the histogram defined by the α-tuple – i.e., $\mathbb{h}(\alpha) = \sum_j \alpha_j \mathbb{h}_j$ – satisfies a classification threshold condition.

Linear system:

$$H\alpha = \mathbb{h}$$

• Determined case:

$B = k$

$\alpha = H^{-1}\mathbb{h}$

(assuming that the k $\mathbb{h}_j$ vectore ar linearly independent)

• Over-determined case:

$B > k$

$\alpha = (H^T H)^{-1} H^T \mathbb{h}$

(least-squares solution defining an approximation with minimal error)

⇒ Consider approximation error:

$e = \| \mathbb{h} - \mathbb{h}(\alpha) \|^2$.

With $\mathbb{d} = \mathbb{h} - \mathbb{h}(\alpha)$, the value of the error is $e = \langle \mathbb{d}, \mathbb{d} \rangle$.

...

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

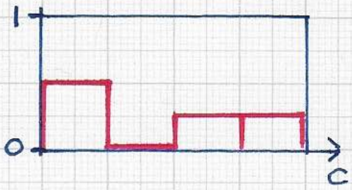
• Laplacian eigenfunctions: ... **Note.** The BEST APPROXIMATION problem to be solved in the overdetermined case is also a CONSTRAINED BEST APPROXIMATION problem. Since the approximation $\mathbb{h}(\alpha)$ must be a "valid histogram," these two conditions must both be satisfied:

$$i) \quad \mathbb{h}(\alpha) \geq \mathbb{0}, \text{ i.e., } h_l(\alpha) \geq 0,\ l=1 \ldots B,$$

$$ii) \quad \sum_{l=1}^{B} h_l(\alpha) = 1 \quad .$$

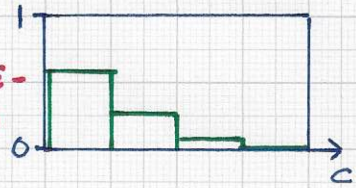
h(S)

S



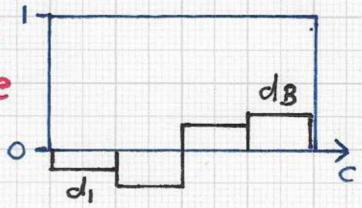
h(α)

best approximation



h(S) − h(α)

difference



Difference of histograms. The histogram of the segment S to be classified is called h(S); the best approximation of h(S) is the histogram h(α) that is the best-possible linear combination of the stored sample histograms $h_j = h(S_j)$, the histograms of sample segments S_j, $j = 1 \ldots k$. The histogram h(α) is the result of solving an over-determined constrained linear system.

• Recall: $\mathbb{d}_j = \mathbb{h} - \mathbb{h}_j$ are the segment-specific histogram differences.

⇒ Define errors $e_j = e(\mathbb{d}_j)$.

Therefore, one must employ a linear system solver that can ensure that these conditions are satisfied when calculating α. The figure (left) shows a simple 4-bin example. Using vector notation for histograms, the given histogram is $\mathbb{h}$ (top), the best approximation is $\mathbb{h}(\alpha)$ (middle) and their difference is $\mathbb{h} - \mathbb{h}(\alpha) = \mathbb{d}$ (bottom). One can define various error measures based on $\mathbb{d} = (d_1, \ldots, d_B)^T$, e.g.:

– $e_{RMS} = (\langle \mathbb{d}, \mathbb{d} \rangle / B)^{1/2}$,

– $e_{MS} = \langle \mathbb{d}, \mathbb{d} \rangle / B$,

– $e_S = \langle \mathbb{d}, \mathbb{d} \rangle$,

– $e_{MAX} = \max \{ |d_l|,\ l = 1 \ldots B \}$.

⇒ Define error $e = e(\mathbb{d})$. ...

■ OBJECT AND MATERIAL EIGENFUNCTIONS — Cont'd.

• Laplacian eigenfunctions: ... We discuss the under-determined case next ($\mathcal{B} < k$). In this case, one must construct a solution — from the many solutions possible — that is in some sense "optimal" and thus preferable.

• Linear system:

$$\mathbb{H}\,\boldsymbol{\alpha} = \mathbb{h}$$

• Under-determined case:

$\mathcal{B} < k$

⇒ Two-step method:

i) Solve

$$\mathbb{H}\mathbb{H}^T \beta_j = \mathbb{h} - \mathbb{H}\hat{\boldsymbol{\alpha}}_j$$

$\Rightarrow \beta_j = \ldots$

ii) Compute

$$\boldsymbol{\alpha}_j = \hat{\boldsymbol{\alpha}}_j + \mathbb{H}^T \beta_j\,,$$

$j = 1 \ldots k$.

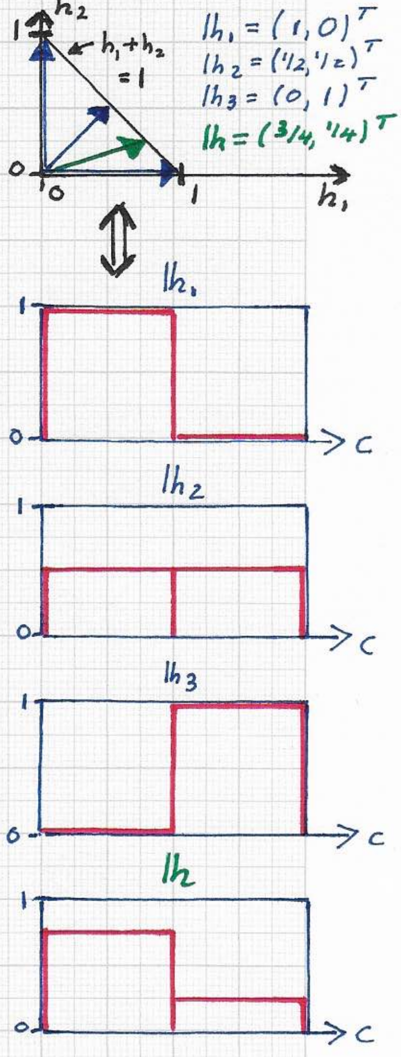
• Explanations:

- The $\hat{\boldsymbol{\alpha}}_j$-tuples are the tuples of "perfect matches," i.e.,

$$\hat{\boldsymbol{\alpha}}_1 = (1, 0, \ldots, 0),$$
$$\vdots$$
$$\hat{\boldsymbol{\alpha}}_k = (0, \ldots, 0, 1).$$

- One computes the k solution tuples $\boldsymbol{\alpha}_j$.
- One must determine which tuple $\boldsymbol{\alpha}_j$ is "optimal" to define $\mathbb{h}(\boldsymbol{\alpha}_j)$.

A simple example is discussed in detail (right). Here, $k=3$ and $\mathcal{B}=2$. An "optimal" representation for $\mathbb{h}$ must be determined to decide whether $\mathbb{h}$ is a new, given histogram that captures histogram behavior indicative of class 1, class 2 or class 3. The general linear algebra solution approach (left) is based on specifying "ideal solution vectors" $\hat{\boldsymbol{\alpha}}_j$ a priori and then computing a "best" solution $\boldsymbol{\alpha}_j$. ...

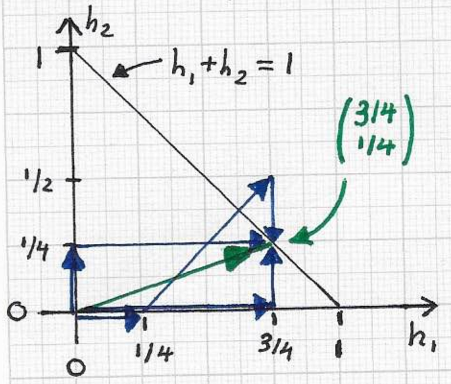


Under-determined case illustrated with a simple example showing data in geometrical "vectorform" and as histograms:

$$\mathbb{h} = \alpha_1 \mathbb{h}_1 + \alpha_2 \mathbb{h}_2 + \alpha_3 \mathbb{h}_3 .$$

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• *Laplacian eigenfunctions:* ... Following the general approach, one must first establish desired "ideal solution vectors" $\hat{\alpha}_j$, $j = 1,2,3$. We define these ideal coefficient tuples $\hat{\alpha}_j$ as tuples representing "ideal match tuples", i.e.,
$\hat{\alpha}_1 = (1,0,0)$, $\hat{\alpha}_2 = (0,1,0)$, $\hat{\alpha}_3 = (0,0,1)$.
These tuples would be ideal tuples if $\mathbb{h}$ were equal to $\mathbb{h}_1$, $\mathbb{h}_2$ or $\mathbb{h}_3$, respectively. Since $\mathbb{h}$ is not equal to one of the sample histograms $\mathbb{h}_i$, we compute three best solutions using the defined $\hat{\alpha}_j$ tuples:



Three best representations for the given vector/histogram $\mathbb{h}$ result, subject to the choices of the values of $\hat{\alpha}_j$, $j = 1,2,3$:

• solution 1:

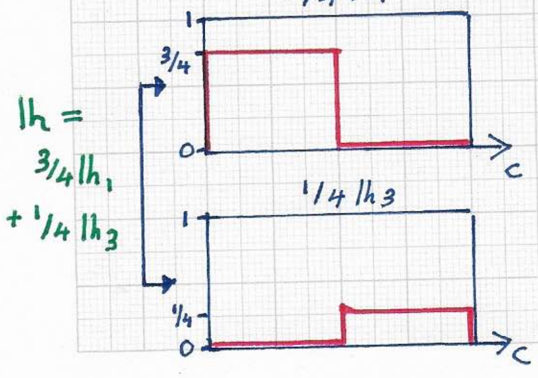
$$\begin{pmatrix}3/4\\1/4\end{pmatrix} = \frac{3}{4}\begin{pmatrix}1\\0\end{pmatrix} + 0\begin{pmatrix}1/2\\1/2\end{pmatrix} + \frac{1}{4}\begin{pmatrix}0\\1\end{pmatrix}$$

• solution 2:

$$\begin{pmatrix}3/4\\1/4\end{pmatrix} = \frac{1}{4}\begin{pmatrix}1\\0\end{pmatrix} + 1\begin{pmatrix}1/2\\1/2\end{pmatrix} - \frac{1}{4}\begin{pmatrix}0\\1\end{pmatrix}$$

• solution 3:

$$\begin{pmatrix}3/4\\1/4\end{pmatrix} = \frac{1}{4}\begin{pmatrix}0\\1\end{pmatrix} + 0\begin{pmatrix}1/2\\1/2\end{pmatrix} + \frac{3}{4}\begin{pmatrix}1\\0\end{pmatrix}$$



Solution 1 visualized in histogram form.

• Given linear system:

$$\alpha_1 \mathbb{h}_1 + \alpha_2 \mathbb{h}_2 + \alpha_3 \mathbb{h}_3 = \begin{pmatrix}3/4\\1/4\end{pmatrix}$$

$$\alpha_1 \begin{pmatrix}h_1^1\\h_2^1\end{pmatrix} + \alpha_2 \begin{pmatrix}h_1^2\\h_2^2\end{pmatrix} + \alpha_3 \begin{pmatrix}h_1^3\\h_2^3\end{pmatrix} = \begin{pmatrix}3/4\\1/4\end{pmatrix}$$

$$\begin{pmatrix}1 & 1/2 & 0\\0 & 1/2 & 1\end{pmatrix}\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix} = \begin{pmatrix}3/4\\1/4\end{pmatrix}$$

$$H \quad \alpha = \begin{pmatrix}3/4\\1/4\end{pmatrix}$$

i) Solve $H H^T \beta_j = \mathbb{h} - H\hat{\alpha}_j$, $j = 1,2,3$:

$$\begin{pmatrix}1 & 1/2 & 0\\0 & 1/2 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\1/2 & 1/2\\0 & 1\end{pmatrix}\begin{pmatrix}\beta_1^j\\\beta_2^j\end{pmatrix} = \begin{pmatrix}3/4\\1/4\end{pmatrix} - \begin{pmatrix}1 & 1/2 & 0\\0 & 1/2 & 1\end{pmatrix}\begin{array}{ccc} j=1 & j=2 & j=3\\ \left(\begin{array}{c|c|c}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{array}\right)\end{array}$$

$$\begin{pmatrix}5/4 & 1/4\\1/4 & 5/4\end{pmatrix}\begin{pmatrix}\beta_1^j\\\beta_2^j\end{pmatrix} = \begin{array}{ccc}\left(\begin{array}{c|c|c}-1/4 & 1/4 & 3/4\\1/4 & -1/4 & -3/4\end{array}\right)\\ j=1 & j=2 & j=3\end{array}$$

...