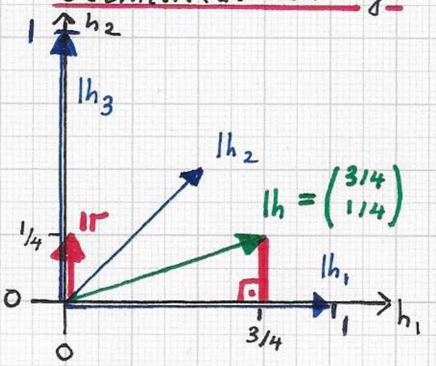
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: ... The resulting three solutions of this linear system for β_j, $j=1,2,3$, are

$$\beta_j = \begin{pmatrix} \beta_1^j \\ \beta_2^j \end{pmatrix} = \left(\begin{array}{c|c|c} -1/4 & 1/4 & 3/4 \\ 1/4 & -1/4 & -3/4 \end{array} \right) .$$

• Geometrical meaning



ii) Calculate the three solutions:

$$\alpha_j = \hat{\alpha}_j + H^T \beta_j \quad , j=1,2,3$$

$$\alpha_j = \left(\begin{array}{c|c|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right) + \begin{pmatrix} 1 & 0 \\ 1/2 & 1/2 \\ 0 & 1 \end{pmatrix} \left(\begin{array}{c|c|c} -1/4 & 1/4 & 3/4 \\ 1/4 & -1/4 & -3/4 \end{array} \right)$$

$$= \left(\begin{array}{c|c|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right) + \left(\begin{array}{c|c|c} -1/4 & 1/4 & 3/4 \\ 0 & 0 & 0 \\ 1/4 & -1/4 & -3/4 \end{array} \right)$$

$$= \left(\begin{array}{c|c|c} 3/4 & 1/4 & 3/4 \\ 0 & 1 & 0 \\ 1/4 & -1/4 & 1/4 \end{array} \right) .$$

• solution 1:

$\hat{\alpha}_1 = (1, 0, 0)$

$\Rightarrow$ project $\mathbb{h}$ onto $\mathbb{h}_1$:

$\Rightarrow 3/4\ \mathbb{h}_1$

$\Rightarrow$ residual is

$\mathbb{r} = \mathbb{h} - 3/4\ \mathbb{h}_1 = \begin{pmatrix} 0 \\ 1/4 \end{pmatrix}$

$\Rightarrow$ express $\mathbb{r}$ via $\mathbb{h}_2, \mathbb{h}_3$:

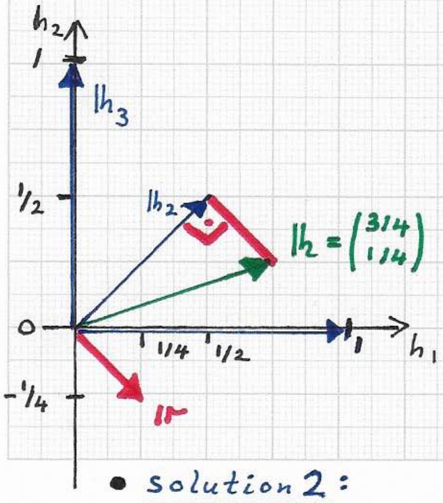
$\mathbb{r} = 0\ \mathbb{h}_2 + 1/4\ \mathbb{h}_3$

$\Rightarrow \alpha_1 = (3/4, 0, 1/4)$

THREE SOLUTIONS result since THREE TUPLES $\hat{\alpha}_j$, $j=1,2,3$, WERE SPECIFIED.

Using vector notation for histograms, the results are THREE WAYS TO EXPAND $\mathbb{h}$:

$$\mathbb{h} = \begin{cases} \sum_{j=1}^{3} \alpha_j^1\ \mathbb{h}_j \\ \sum_{j=1}^{3} \alpha_j^2\ \mathbb{h}_j \\ \sum_{j=1}^{3} \alpha_j^3\ \mathbb{h}_j \end{cases} ,$$



where the three coefficient tuples are

$$\alpha_1 = (3/4, 0, 1/4) \quad , \alpha_1 = (\alpha_1^1, \alpha_2^1, \alpha_3^1),$$

$$\alpha_2 = (1/4, 1, -1/4) \quad , \alpha_2 = (\alpha_1^2, \alpha_2^2, \alpha_3^2),$$

$$\alpha_3 = (3/4, 0, 1/4) \quad , \alpha_3 = (\alpha_1^3, \alpha_2^3, \alpha_3^3) .$$

• solution 2:

$\hat{\alpha}_2 = (0, 1, 0)$

...

The chosen solution approach for the under-determined system identifies these three possibilities to expand $\mathbb{h}$ via $\{\mathbb{h}_j\}_{j=1}^{3}$ as the "best possibilities."

...

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: ...

• Geometrical meaning

• solution 2: ...

$\Rightarrow$ project $\mathbf{h}$ onto $\mathbf{h}_2$:

$\Rightarrow 1\,\mathbf{h}_2$

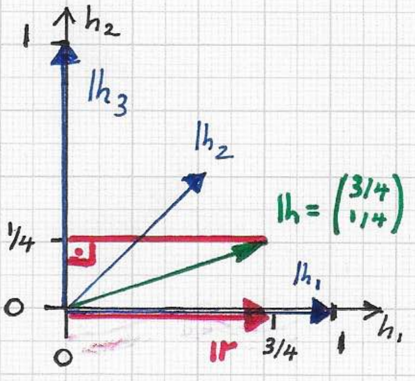
$\Rightarrow$ residual is

$\mathbf{r} = \mathbf{h} - 1\,\mathbf{h}_2 = \begin{pmatrix} 1/4 \\ -1/4 \end{pmatrix}$

$\Rightarrow$ express $\mathbf{r}$ via $\mathbf{h}_1, \mathbf{h}_3$:

$\mathbf{r} = 1/4\,\mathbf{h}_1 - 1/4\,\mathbf{h}_3$

$\Rightarrow \boldsymbol{\alpha}_2 = (1/4, 1, -1/4)$



• solution 3:

$\hat{\boldsymbol{\alpha}}_3 = (0, 0, 1)$

$\Rightarrow$ project $\mathbf{h}$ onto $\mathbf{h}_3$:

$\Rightarrow 1/4\,\mathbf{h}_3$

$\Rightarrow$ residual is

$\mathbf{r} = \mathbf{h} - 1/4\,\mathbf{h}_3 = \begin{pmatrix} 3/4 \\ 0 \end{pmatrix}$

$\Rightarrow$ express $\mathbf{r}$ via $\mathbf{h}_1, \mathbf{h}_2$:

$\mathbf{r} = 3/4\,\mathbf{h}_1 + 0\,\mathbf{h}_2$

$\Rightarrow \boldsymbol{\alpha}_3 = (3/4, 0, 1/4)$

... It is important to consider the geometrical meaning of this specific method for calculating solutions for an underdetermined linear equation system. The previous page and this page provide illustrations that explain the geometrical derivation of solutions (left):

→ The given vector $\mathbf{h}$ (in the plane) must be expanded via vectors $\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3$.

→ A "desired solution $\hat{\boldsymbol{\alpha}}_j$-tuple" – with $\hat{\boldsymbol{\alpha}}_1 = (1,0,0)$, $\hat{\boldsymbol{\alpha}}_2 = (0,1,0)$ and $\hat{\boldsymbol{\alpha}}_3 = (0,0,1)$ – forces an initial perpendicular projection of $\mathbf{h}$ onto vector $\mathbf{h}_j$.

→ The vector resulting from this projection onto $\mathbf{h}_j$ is a vector $\mathbf{h}_p$; a "residual vector $\mathbf{r}$" can now be defined as $\mathbf{r} = \mathbf{h} - \mathbf{h}_p$.

→ The residual vector $\mathbf{r}$ is represented, uniquely, via a linear combination of the other two vectors, i.e., via $\mathbf{h}_{j_2}$ and $\mathbf{h}_{j_2}$, $j, j_1, j_2 \in \{1,2,3\}$.

→ For each tuple $\hat{\boldsymbol{\alpha}}_j$ one obtains a "best solution tuple $\boldsymbol{\alpha}_j$" that defines an exact representation of $\mathbf{h}$ via a linear combination of "sample vectors $\mathbf{h}_j$."

...

■ OBJECT AND MATERIAL EIGENFUNCTIONS — Cont'd.

• Laplacian eigenfunctions: ... Generally, one cannot determine a priori, i.e., prior to actual practical use, whether a material classification problem will lead to the over-determined case ($B > k$), the determined case ($B = k$) or the under-determined case ($B < k$). The classification goal for a new, given data segment is to make a decision whether the segment belongs to a material class for which multiple (multi-scale) coefficient, discrete histograms are known (via "training" with several class-associated samples). The classification approach must handle the three cases — $B > k$, $B = k$ and $B < k$ — "equally well." The fundamental problem to solve concerns the determination of a "match": Does the multi-scale coefficient histogram characterization of a new, given segment match (or satisfy an allowable distance threshold to) that of a material used for "training"?

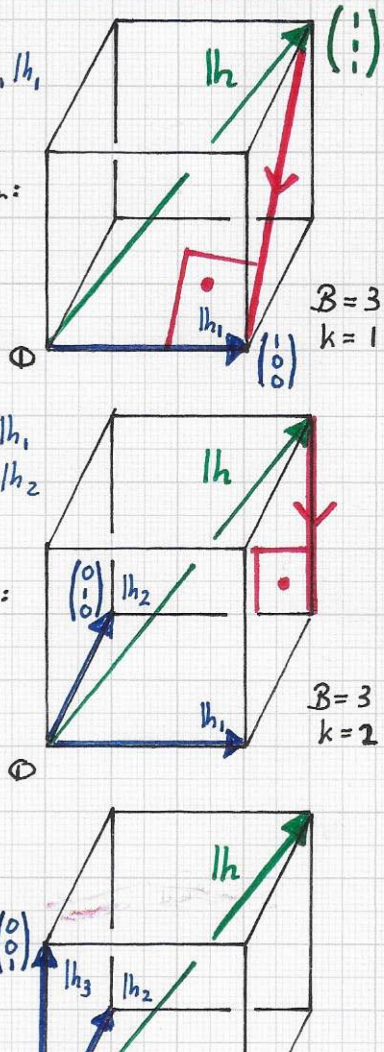
The figure (left) illustrates that the best representation/approximation of $\mathbb{h}$ for $B \geq k$ is a perpendicular projection of $\mathbb{h}$ onto the sub-space spanned by $\mathbb{h}_1, \mathbb{h}_2, \ldots$ ...

• Over-determined case ($B \geq k$) — geometry:

system: $\mathbb{h} = \alpha_1 \mathbb{h}_1$
⇒ least-squares solution: $\mathbb{h}^a$

system: $\mathbb{h} = \alpha_1 \mathbb{h}_1 + \alpha_2 \mathbb{h}_2$
⇒ least-squares solution: $\mathbb{h}^a$

system: $\mathbb{h} = \alpha_1 \mathbb{h}_1 + \alpha_2 \mathbb{h}_2 + \alpha_3 \mathbb{h}_3$
⇒ exact solution



Solving the resulting NORMAL EQUATIONS for these three cases leads to the three BEST APPROXIMATIONS

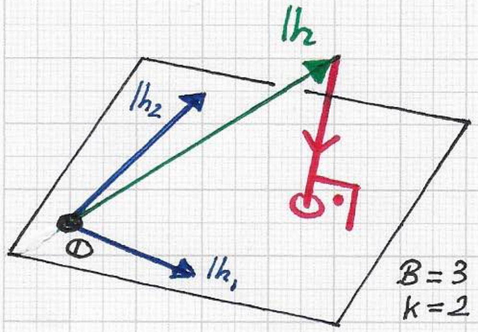
$\mathbb{h}^a = \sqrt{3}/3 \; \mathbb{h}_1$,
$\mathbb{h}^a = \sqrt{3}/3 \; (\mathbb{h}_1 + \mathbb{h}_2)$ and
$\mathbb{h} = \sqrt{3}/3 \; (\mathbb{h}_1 + \mathbb{h}_2 + \mathbb{h}_3)$.

(For $k = 1, 2$ the solutions are perpendicular projections of $\mathbb{h}$ onto 1D/2D spaces.)

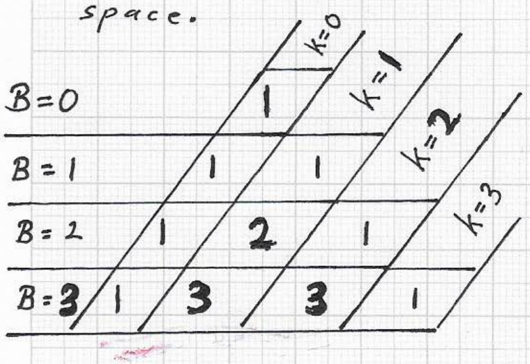
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: …

• Over-determined case ($B>k$) – geometry:



Best approximation of lh understood as perpendicular projection onto the 2D sub-space spanned by lh_1 and lh_2. All vectors are vectors in 3D space.



Pascal's triangle showing the number of possibilities to project a "histogram vector lh" in a space of dimension B onto a sub-space of dimension k when the sub-space can be spanned by k "histogram vectors lh_i, $i=0,1,\ldots,B$, that are also (linearly independent) vectors in a space of dimension B.

One can consider the following "combinatorial possibilities": If one had three linearly independent "histogram vectors" lh_1, lh_2 and lh_3, considering projections onto three 1D sub-spaces (defined by lh_1, by lh_2 and lh_3) and onto three 2D sub-spaces (defined by lh_1 and lh_2, by lh_1 and lh_3 and by lh_2 and lh_3) would be possible. Generally, when lh is a vector in a space of dimension B and B linearly independent vectors lh_i exist – themselves lying in the space of dimension B – one can use all possible subsets of $\{lh_i\}_{i=1}^{B}$ (spanning sub-spaces of dimension $0,1,2,\ldots,B$) to compute best approximations of lh in these sub-spaces. The number of possibilities is: $\sum_{k=0}^{B}\binom{B}{k}=2^B$. Since only a small number of these possibilities can be considered, one must limit the number of vectors lh_i selected to define sub-spaces of only low dimensions, e.g., only sub-spaces of dimensions 1, 2 or 3.

…

Bernd Hamann
Stratovan

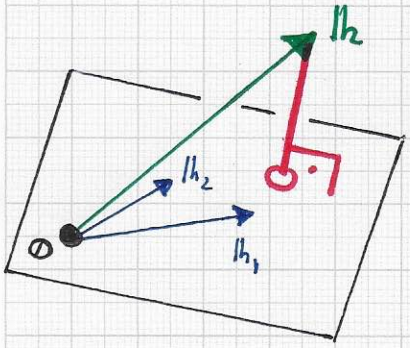
12/31/2021

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions: ... **Note.** Recall that the new, given "histogram vector $\mathbb{h}$" should either be very similar to one of the stored sample "histogram vectors $\mathbb{h}_i$" – indicating a likely match – or would require a relatively large number of sample vectors $\mathbb{h}_i$ to arrive at a "good best approximation" of $\mathbb{h}$. Thus, limiting the number of dimensions of the sub-spaces to project $\mathbb{h}$ onto can be done for making the decision about match or mis-match. Having to use a relatively large number of vectors $\mathbb{h}_i$ to obtain an acceptable best approximation would indicate a likely mis-match, and higher-dimensional sub-spaces do therefore not need to be used for projection. In summary, one could use the best approximation method in the over-determined case in the following way:

- → Project $\mathbb{h}$ onto the **1D** sub-spaces of $\mathbb{h}_1$; ...; of $\mathbb{h}_k$.
- → Project $\mathbb{h}$ onto the **2D** sub-spaces of all pairs of sample vectors, i.e., $\mathbb{h}_1$ and $\mathbb{h}_2$; ...; $\mathbb{h}_{k-1}$ and $\mathbb{h}_k$.
- → Project $\mathbb{h}$ onto the **3D** sub-spaces of all triples of sample vectors, i.e., $\mathbb{h}_1$, $\mathbb{h}_2$ and $\mathbb{h}_3$; ...; $\mathbb{h}_{k-2}$, $\mathbb{h}_{k-1}$ and $\mathbb{h}_k$. ...

Eventually, a decision about "match or not" must be made. ...



In the general geometrical case, the linearly independent vectors $\mathbb{h}_1$ and $\mathbb{h}_2$ are not normalized and are not orthogonal to each other. The vector $\mathbb{h}$ is also an arbitrary vector. In this example the best perpendicular projection of $\mathbb{h}$ onto the 2D subspace defined by $\mathbb{h}_1$ and $\mathbb{h}_2$ in 3D space is given by the normal equations

$$\begin{bmatrix} \langle \mathbb{h}_1, \mathbb{h}_1 \rangle & \langle \mathbb{h}_1, \mathbb{h}_2 \rangle \\ \langle \mathbb{h}_2, \mathbb{h}_1 \rangle & \langle \mathbb{h}_2, \mathbb{h}_2 \rangle \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} \langle \mathbb{h}, \mathbb{h}_1 \rangle \\ \langle \mathbb{h}, \mathbb{h}_2 \rangle \end{bmatrix}.$$

The solution is the best possible approximation of $\mathbb{h}$ in the plane spanned by $\mathbb{h}_1$ and $\mathbb{h}_2$, i.e.,

$$\alpha_1 \mathbb{h}_1 + \alpha_2 \mathbb{h}_2$$

minimizes the residual vector $\mathbb{r}$, with

$$\mathbb{r} = \mathbb{h} - (\alpha_1 \mathbb{h}_1 + \alpha_2 \mathbb{h}_2),$$

in the least-squares sense.