Bernd Hamann
Stratovan

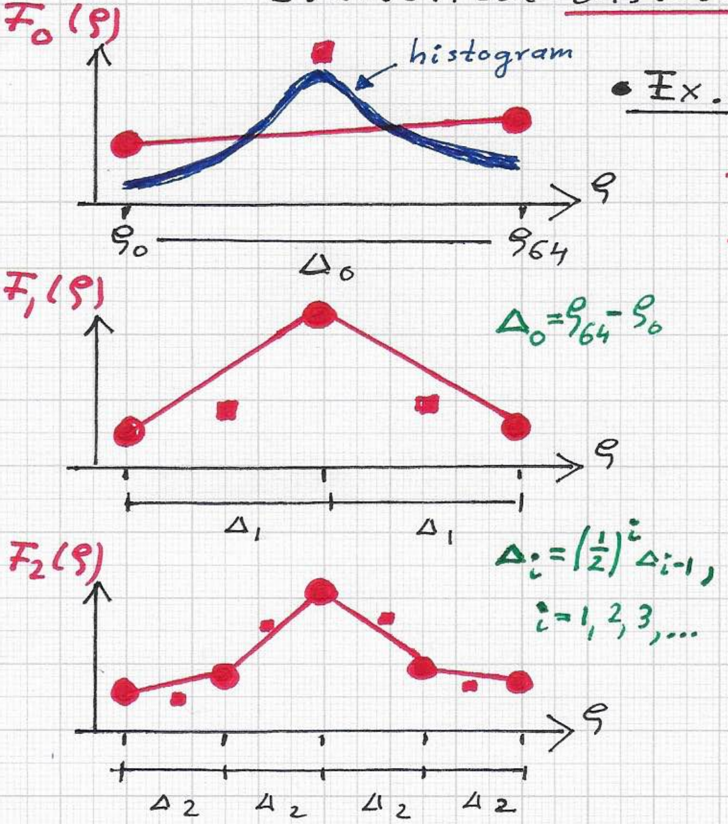




## ■ RICHARDSON EXTRAPOLATION & BEST APPROXIMATION

- GOAL: Statistical characteristics of a distribution function – e.g., a spline-based density function approximating a histogram optimally – should be "as precise as possible." RICHARDSON EXTRAPOLATION is one (appropriate and efficient) method for estimating HIGH-PRECISION statistical distribution characteristics.

REPEATED BISECTION




original "histogram function" (e.g., 64 bins)

segment of best approx. Linear spline

●, ■ current & next spline segment end points

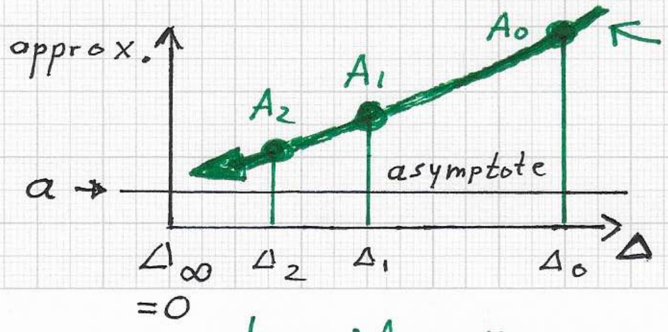
- Ex.: Compute a sequence of best linear spline approximations for a histogram of a segment's density ($\rho$)-values. The numbers of spline segments are 1, 2, 4, 8, ..., and the individual spline segments are defined over $\rho$-intervals of width $\Delta_0, \Delta_1, \Delta_2, \Delta_3, \ldots$; the spline sequence is $F_0(\rho)$, $F_1(\rho)$, $F_2(\rho)$, ... The sequence of errors $E_0, E_1, E_3, \ldots$ associated with these splines produces decreasing error values.

$\Rightarrow (\mu, \sigma^2)$ sequence improves!

## ■ RICHARDSON EXTRAPOLATION – Cont'd.

- **Ex.:** Considering the above example, the best linear splines $F_0(\varrho)$, $F_1(\varrho)$, $F_3(\varrho)$, ... are (normalized) probability density functions. This sequence of $F_i(\varrho)$ splines (and the original binned histogram data) is finite. WHILE EACH SPLINE $F_i(\varrho)$ HAS AN ASSOCIATED SET OF MAIN STATISTICAL CHARACTERISTICS, e.g., $\mu$- and $\sigma^2$-values, HOW CAN ONE USE "EXTRAPOLATION TO THE LIMIT $\Delta_\infty = 0$" TO COMPUTE BEST-POSSIBLE VALUES OF THE STATISTICAL CHARACTERISTICS? In support of classification... We consider the case of performing extrapolation of the mean $\mu$.

- **Ex.:** The best-approximating splines $F_0(\varrho)$, $F_1(\varrho)$, $F_2(\varrho)$, ... with associated spacing $\Delta_0$, $\Delta_1$, $\Delta_2$, ... have mean values $\mu_0$, $\mu_1$, $\mu_2$, ... ITERATED RICHARDSON EXTRAPOLATION produces sequences of mean values by extrapolation to the limit case $\Delta_\infty = 0$.




$A(\Delta)$ is an asymptotic expansion of order $p$:

$A(\Delta) = a + c_p \Delta^p (+\ldots)$,

$A_i = A(\Delta_i)$,

$\Delta_{i+1} = \alpha \Delta_i$ with $0 < \alpha < 1$

here: $A_i = \mu_i$

"$A_\infty$ = 'best' $\mu$-value"

## RICHARDSON EXTRAPOLATION – Cont'd.

- Summary of Approach: Compute a sequence of "improving approximations" $A_i$ and use this sequence to estimate extrapolated values for the limit case $\Delta_\infty = 0$; use the $A_i$-values to compute additional sequences that converge to the limit value $A_\infty$ even more quickly...

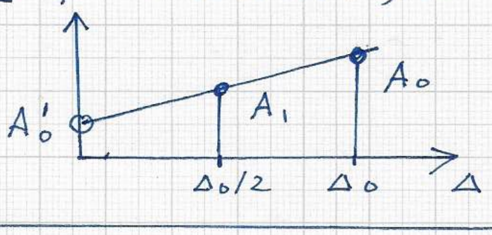
- $A_0 = a + c_p \Delta_0^p \quad \Rightarrow \quad c_p = (A_0 - a)/\Delta_0^p$

  $A_1 = a + c_p \Delta_1^p \quad = a + (A_0 - a)/\Delta_0^p \cdot (\alpha \Delta_0)^p$

  $\quad = (1-\alpha^p)\, a + A_0 \alpha^p$

  $\Rightarrow \quad a = (A_1 - A_0 \alpha^p)/(1-\alpha^p) = A_0^1$

- Ex.: $p = 1$ (linear), $\alpha = 1/2$ :




$A_0^1 = (A_1 - \frac{1}{2} A_0)/ {}^1\!/_2$

$\quad = 2A_1 - A_0$

- General formula for $A_i^1$ : [set $A_i^0 = A_i$]

$$A_0^1 = (A_1^0 - A_0^0 \alpha^p)/(1-\alpha^p)$$

$$A_1^1 = (A_2^0 - A_1^0 \alpha^p)/(1-\alpha^p)$$

$$\Rightarrow \quad A_i^1 = (A_{i+1}^0 - A_i^0 \alpha^p)/(1-\alpha^p)$$

"2nd sequence of extrapolated data"

## RICHARDSON EXTRAPOLATION – Cont'd.

- Perform "repeated extrapolation" – assuming that the asymptotic expansion of $A(\Delta)$ is of the form

$$A(\Delta) = a + c_p \Delta^p + c_{2p}\Delta^{2p} + \ldots + c_{kp}\Delta^{kp} \; (+\ldots)$$

- $$A_0^1 = a + c_{2p}\Delta_0^{2p} \quad \Rightarrow \quad c_{2p} = (A_0^1 - a)/\Delta_0^{2p}$$

$$A_1^1 = a + c_{2p}\Delta_1^{2p} = a + (A_0^1 - a)/\Delta_0^{2p} \cdot (\alpha\Delta_0)^{2p}$$
$$= (1-\alpha^{2p})\,a + A_0^1\alpha^{2p}$$

$$\Rightarrow \quad a = (A_1^1 - A_0^1\alpha^{2p})/(1-\alpha^{2p}) = A_0^2$$

- General formula for $A_i^2$:

$$A_i^2 = (A_{i+1}^1 - A_i^1\alpha^{2p})/(1-\alpha^{2p})$$

- **GENERAL SCHEME:**

$$\begin{tikzcd}
i=0 & A_0^0 \arrow[r] & A_0^1 \arrow[r] & A_0^2 \arrow[r] & A_0^3 \\
i=1 & A_1^0 \arrow[r] \arrow[ur] & A_1^1 \arrow[r] \arrow[ur] & A_1^2 \arrow[ur] & \\
i=2 & A_2^0 \arrow[r] \arrow[ur] & A_2^1 \arrow[ur] & & \\
i=3 & A_3^0 \arrow[ur] & & & \\
 & j=0 & j=1 & j=2 & j=3
\end{tikzcd}$$

FORMULA:

$$\boxed{A_i^j = \frac{A_{i+1}^{j-1} - A_i^{j-1}\alpha^{jp}}{1-\alpha^{jp}}} \quad (*)$$

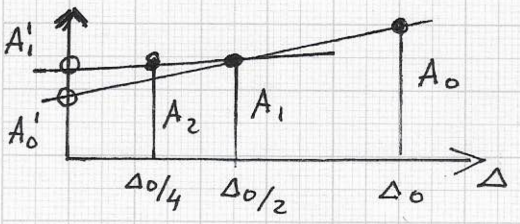
- Algorithm:
  - compute $A_i^0 = A_i$, $i = 0 \ldots N$
  - for $(j=1)$ to $(j=N)$ do

    for $(i=0)$ to $(N-j)$ do

    compute $A_i^j = \ldots$ $(*)$
  - return $A_0^N$

## ■ RICHARDSON EXTRAPOLATION – Cont'd.

● Ex.: Linear case ($p=1$) and bisection ($\alpha = 1/2$)




→ $A_i^0 = A_i$ (= approximations based on $\Delta_i$)

→ $A_i^1 = 2A_{i+1}^0 - A_i^0$ (see p. 11)

→ $$A_i^j = \frac{A_{i+1}^{j-1} - A_i^{j-1}/2^j}{1 - 1/2^j}$$

● USE: Repeated extrapolation of mean ($\mu$) values:

$$\begin{array}{llll} \mu_0^0 & \mu_0^1 & \mu_0^2 & \cdots \; \mu_0^N \\ \mu_1^0 & \mu_1^1 & \mu_1^2 & \\ \mu_2^0 & \mu_2^1 & \vdots & \\ \vdots & \vdots & & \\ \mu_N^0 & & & \end{array}$$

$N \to \infty$

$\Rightarrow \mu_0^\infty$ "ideal"

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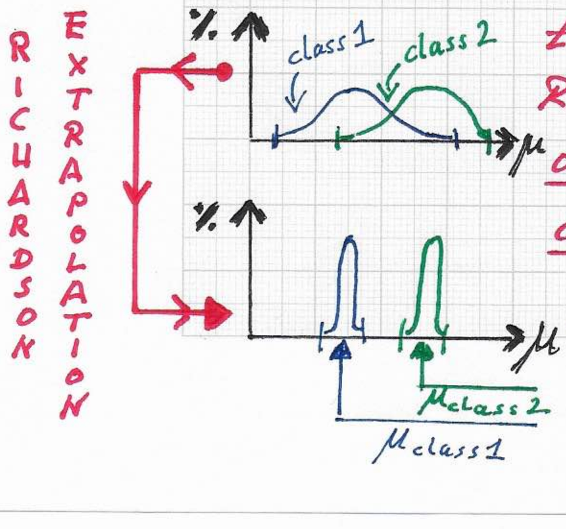
● WHY? → By improving statistical characteristics of a material class (or even multiple segments/objects belonging to the same material class) through Richardson extrapolation, separation of multi-class data and 'best-possible' classification is supported.

2-class example:




⟸ Richardson extrapolation "focuses" the mean values $\mu_{class1}$ and $\mu_{class2}$ and can eliminate overlap of these two means' distribution. Unique classification is possible. ☺ BH