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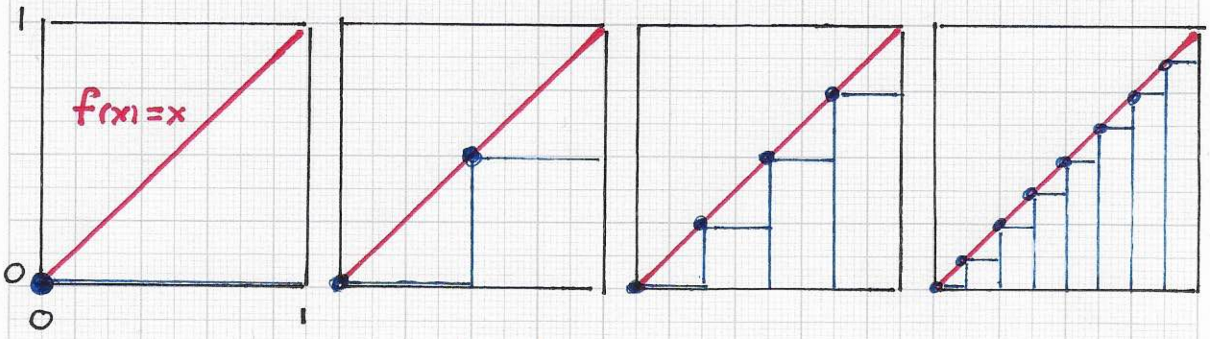
■ RICHARDSON EXTRAPOLATION – EXAMPLES

- Integral computation of a linear and quadratic function demonstrate the RAPID CONVERGENCE of Richardson extrapolation – even when using initially a piecewise constant integral estimator

1) Integrating $f(x) = x$, $x \in [0, 1]$: $\int_0^1 x\,dx = \frac{1}{2}$

$i = 0$ $\qquad$ $i = 1$ $\qquad$ $i = 2$ $\qquad$ $i = 3$



$\Delta_0 = 1$ $\qquad$ $\Delta_1 = \frac{\Delta_0}{2} = \frac{1}{2}$ $\qquad$ $\Delta_2 = \frac{1}{4}$ $\qquad$ $\Delta_3 = \frac{1}{8}$

$\Rightarrow$ sequence of integral value approximations I_i:

$I_0 = 0 = \frac{0}{2}$ $\qquad$ $I_1 = \frac{1}{4}$ $\qquad$ $I_2 = \frac{3}{8}$ $\qquad$ $I_3 = \frac{7}{16}$

$$\Rightarrow I_i^0 = I_i = \frac{2^i - 1}{2^{i+1}}$$

$\Rightarrow$ Richardson extrapolation : $I_i^j = \dfrac{I_{i+1}^{j-1} - I_i^{j-1} \cdot \alpha^{jp}}{1 - \alpha^{jp}}$

	$j = 0$		$j = 1$
$i=0$	$I_0^0 = \frac{0}{2}$	$\xrightarrow{\cdot(-1)}$, $\cdot 2$	$I_0^1 = \frac{1}{2}$ ← done!
$i=1$	$I_1^0 = \frac{1}{4}$	$\xrightarrow{\cdot(-1)}$, $\cdot 2$	$I_1^1 = \frac{1}{2}$
$i=2$	$I_2^0 = \frac{3}{8}$	$\xrightarrow{\cdot(-1)}$, $\cdot 2$	$I_2^1 = \frac{1}{2}$
$i=3$	$I_3^0 = \frac{7}{16}$		

$$\Rightarrow I_i^1 = 2 I_{i+1}^0 - I_i^0$$

where $\alpha = \frac{1}{2}$, $p = 1$

$\Rightarrow$ **CONVERGENCE FOR $j = 1$**

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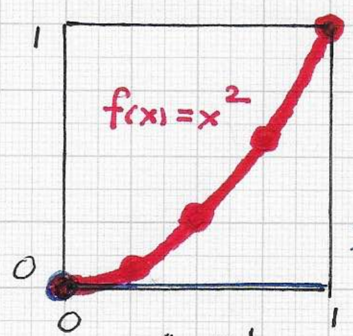
■ RICHARDSON EXTRAPOLATION - EXAMPLES

2) Integrating $f(x) = x^2$, $x \in [0,1]$: $\int_0^1 x^2 dx = \frac{1}{3}$

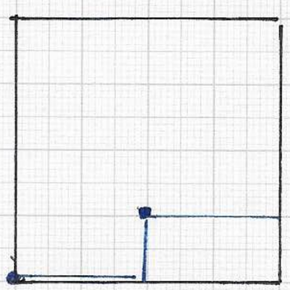
$i=0$



$\Delta_0 = 1$

$I_0 = 0$

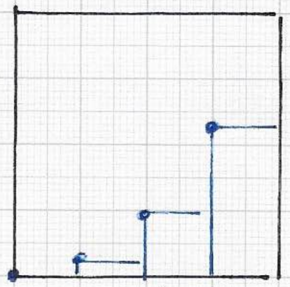
$i=1$



$\Delta_1 = 1/2$

$I_1 = \frac{1}{8}$

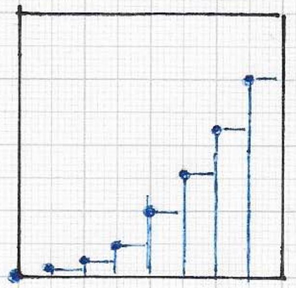
$i=2$



$\Delta_2 = 1/4$

$I_2 = \frac{7}{32}$

$i=3$



$\Delta_3 = 1/8$

$I_3 = \frac{35}{128}$

$\Rightarrow$ sequence of approximations I_i:

- e.g., consider $I_3^0 = I_3$:

$$I_3 = \frac{\Delta_0}{8}\left(\left(\tfrac{0}{8}\Delta_0\right)^2 + \left(\tfrac{1}{8}\Delta_0\right)^2 + \dots + \left(\tfrac{7}{8}\Delta_0\right)^2 \right)$$

$$= \frac{\Delta_0}{8}\left(\frac{\Delta_0}{8}\right)^2 (0+1+4+9+\dots+49)$$

$$= \left(\frac{\Delta_0}{8}\right)^3 \sum_{j=0}^{7} j^2$$

- general I_i^0 computation:

$$\underline{I_i^0} = \left(\frac{\Delta_0}{2^i}\right)^3 \sum_{j=0}^{2^i-1} j^2$$

$\left\{ \underline{\text{note}}: \sum_{j=0}^{N} j^2 = \frac{2N+1}{3} \cdot \frac{N(N+1)}{2} \right\}$

$$= \dots$$

$$= \frac{\Delta_0^3}{3}\left(1 - \frac{1}{2^i} - \frac{1}{2^{i+1}} + \frac{1}{2^{2i+1}}\right)$$

$$\overset{\Delta_0=1}{=} \frac{2^{2i+1} - 2^{i+1} - 2^i + 1}{3 \cdot 2^{2i+1}}$$

$\Rightarrow$ Richardson extrapolation:

$I_0^0 = 0$	$\cdot(-1)$, $\cdot 2$	$I_0^1 = \frac{1}{4}$	$\cdot(-\frac{1}{3})$, $\cdot\frac{4}{3}$	$I_0^2 = \frac{1}{3}$ done
$I_1^0 = \frac{1}{8}$	$\cdot(-1)$, $\cdot 2$	$I_1^1 = \frac{5}{16}$		$I_1^2 = \frac{1}{3}$
$I_2^0 = \frac{7}{32}$	$\cdot(-1)$, $\cdot 2$	$I_2^1 = \frac{21}{64}$		$j=2$
$I_3^0 = \frac{35}{128}$		$j=1$		
$j=0$				

$\Rightarrow$ **CONVERGENCE FOR $j=2$**

$I_i^j = \dots \Rightarrow \underline{I_i^2 = \frac{4I_{i+1}^1 - I_i^1}{3}}$

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■ RICHARDSON EXTRAPOLATION – Cont'd.

(Additional notes regarding this "super-efficient method" for extrapolation)

1) The "repeated extrapolation scheme":

$$A_i^j = \frac{A_{i+1}^{j-1} - A_i^{j-1}\alpha^{jp}}{1-\alpha^{jp}}$$

- Approximation A_i^j is defined as a "combination" of two previous approximations A_{i+1}^{j-1} and A_i^{j-1}, considering a specific stepsize factor α and power p

$\Downarrow$ $\alpha = \frac{1}{2}$ (halfing stepsize)
$p = 1$ (linear polynomial)

$$A_i^j = \frac{2^j A_{i+1}^{j-1} - A_i^{j-1}}{2^{j-1}}$$

	$j=0$	$j=1$	$j=2$	$j=3$	
$i=0$	A_0^0	A_0^1	A_0^2	A_0^3	...
$i=1$	A_1^0	A_1^1	A_1^2	$\vdots$	
$i=2$	A_2^0	A_2^1	$\vdots$		
$i=3$	A_3^0	$\vdots$			
	$\vdots$				

$$A_i^1 = \frac{2A_{i+1}^0 - A_i^0}{1}$$

$$A_i^2 = \frac{4A_{i+1}^1 - A_i^1}{3}$$

$$A_i^3 = \frac{8A_{i+1}^2 - A_i^2}{7}$$

- Basic idea used for A_i^0 computation: Consider the polynomial

$$A(\Delta) = a_0 + a_p\Delta^p$$

and determine its 2 unknown coefficients a_0 and a_p from 2 equations:

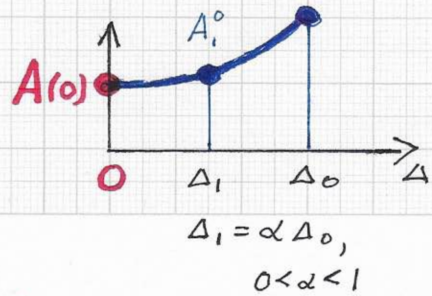
(i) $a_0 + a_p\Delta_0^p = A_0^0$

(ii) $a_0 + a_p(\alpha\Delta_0)^p = A_1^0$

(i) $\Rightarrow a_p = (A_0^0 - a_0)/\Delta_0^p$

insert into (ii):

$$a_0 = \frac{A_1^0 - A_0^0\alpha^p}{1-\alpha^p}$$



$\Delta_1 = \alpha\Delta_0$, $0<\alpha<1$

desired value : $A(0) = a_0$

RICHARDSON EXTRAPOLATION - Cont'd.

2) Improving efficiency by explicit representation of extrapolated values - instead of performing all computations involved in repeated extrapolation

- Given: approximations A_i^0, $i = 0 \ldots N$, generated with corresponding values Δ_i, $\Delta_{i+1} = \frac{1}{2}\Delta_i$, $i = 0 \ldots N-1$ (with $\Delta_0 = \Delta$)

- For this special case (halfing Δ_i-values in each step), the resulting values of A_0^j are:

$$A_0^1 = (2A_1^0 - A_0^0)/1 = \sum_{i=0}^{1} c_i^1 A_i^0 / D_1$$

$$A_0^2 = (8A_2^0 - 6A_1^0 + A_0^0)/3 = \sum_{i=0}^{2} c_i^2 A_i^0 / D_2$$

$$A_0^3 = (64A_3^0 - 56A_2^0 + 14A_1^0 - A_0^0)/21 = \sum_{i=0}^{3} c_i^3 A_i^0 / D_3$$

$$A_0^4 = (1024A_4^0 - 960A_3^0 + 280A_2^0 - 30 + A_0^0 / 315 = \sum_{i=0}^{4} c_i^4 A_i^0 / D_4$$

$$A_0^5 = (32768A_5^0 - 31744A_4^0 + 9920A_3^0 - 1240A_2^0 + 62A_1^0 - A_0^0)/9765 = \sum_{i=0}^{5} c_i^5 A_i^0 / D_5$$

$j \downarrow$

? • **Questions: (i) $A_0^N = ?$, (ii) $\lim_{N \to \infty} A_0^N = ?$**

! • Concerning the general formula for A_0^j, there are 3 facts: → The terms in A_0^j have alternating signs $+, -, +, -, +, - \ldots$

→ The denominator of A_0^j is

$$1 \cdot 3 \cdot 7 \cdot 15 \cdot 31 \cdot \ldots = (2^1-1)(2^2-1)(2^3-1)(2^4-1)(2^5-1)\ldots = \prod_{i=1}^{j}(2^i - 1) = D_j.$$

→ The sum of coefficients c_i^j equals D_j: $\sum_{i=0}^{j} c_i^j = D_j$.

■ RICHARDSON EXTRAPOLATION – Cont'd.

2) ... Formula for $A_0^N = ?$

- One can write the numerator of the expression for A_0^N as a "product of a row of coefficients and a column of A_i^0 values", e.g., $A_0^2 = (8, -6, 1)\begin{pmatrix} A_2^0 \\ A_1^0 \\ A_0^0 \end{pmatrix} / 3.$

$$\Rightarrow A_0^N = \left(c_N^N A_N^0 + c_{N-1}^N A_{N-1}^0 + c_{N-2}^N A_{N-2}^0 + \ldots + c_0^N A_0^0 \right) / D_N$$

$$= \underbrace{\left(c_N^N, c_{N-1}^N, c_{N-2}^N, \ldots, c_0^N \right)}_{=\mathbb{c}^N} \begin{pmatrix} A_N^0 \\ A_{N-1}^0 \\ A_{N-2}^0 \\ \vdots \\ A_0^0 \end{pmatrix} / D_N .$$

$c_i^N = ?$

- Coefficients c_i^N = "Linear combinations of powers of 2": Consider coefficient vectors $\mathbb{c}^1, \mathbb{c}^2, \mathbb{c}^3, \ldots$ written as rows:

$$\begin{aligned}
A_0^1 &: \mathbb{c}^1 = (2, -1) \\
A_0^2 &: \mathbb{c}^2 = (8, -6, 1) \\
A_0^3 &: \mathbb{c}^3 = (64, -56, 14, -1) \\
A_0^4 &: \mathbb{c}^4 = (1024, -960, 280, -30, 1) \\
A_0^5 &: \mathbb{c}^5 = (32768, -31744, 9920, -1240, 62, -1)
\end{aligned}$$

- One can represent these coefficient values in BASE-2 notation:

$$\mathbb{c}^1 = (2, -1)_{10} = (2^1, -2^0)_{10} = (10, -1)_2$$

$$\mathbb{c}^2 = (8, -6, 1)_{10} = (2^3, -2^2 - 2^1, 2^0)_{10} = (1000, -110, 1)_2$$

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■ RICHARDSON EXTRAPOLATION – Cont'd.

2) ... $A_0^N = ?$ Towards a closed-form formula...

$\Rightarrow$ Table of coefficients of 'A_0^N numerators':

		2^{15}	2^{14}	2^{13}	2^{12}	2^{11}	2^{10}	2^9	2^8	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0	
$N=1$	A_0^0																-1	
	A_1^0															1		
$N=2$	A_0^0																$+1$	
	A_1^0														-1	-1		
	A_2^0													1				
$N=3$	A_0^0																-1	
	A_1^0													1	1	1		
	A_2^0											-1	-1	-1				
	A_3^0										1							
$N=4$	A_0^0																$+1$	
	A_1^0												-1	-1	-1	-1		
	A_2^0								$_1$	1_0	1_0	2_0	1_1	1_1				c_2^4
	A_3^0							-1	-1	-1	-1							
	A_4^0						1											
$N=5$	A_0^0																-1	
	A_1^0											1	1	1	1	1		
	A_2^0						$_{-1}$	-1_0	-1_0	-2_{-1}	-2_{-1}	-2_0	-1_{-1}	-1_{-1}				c_2^5
	A_3^0			$_1$	1_0	1_0	2_1	2_1	2_0	1_1	1_1							
	A_4^0		-1	-1	-1	-1	-1											
	A_5^0	1																

$\Rightarrow$ e.g., $c_2^4 = 2^7 + 2^6 + 2\cdot 2^5 + 2^4 + 2^3 = 280 = 2^8 + 2^4 + 2^3$

$c_2^5 = -(2^9 + 2^8 + 2\cdot 2^7 + 2\cdot 2^6 + 2\cdot 2^5 + 2^4 + 2^3) = -1240$

$= -(2^{10} + 2^7 + 2^6 + 2^4 + 2^3)$