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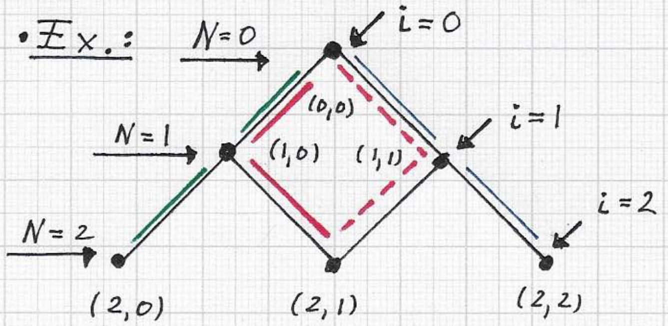
■ RICHARDSON EXTRAPOLATION - Cont'd.

2)... A_0^N = "**closed-form formula**"

$\Rightarrow$ GEORGE POLYA described the correspondence between BINOMIAL COEFFICIENTS and the number of all possible paths from the root of PASCAL'S TRIANGLE to a leaf node via BLOCK-WALKING. (See: Alan Tucker, "Applied Combinatorics", pp. 203–207.)

• Ex.:



• No. paths connecting root (0,0) and leaf node (N,i) is $\binom{N}{i}$.

• here: $\binom{2}{0} = 1$, $\binom{2}{1} = 2$, $\binom{2}{2} = 1$

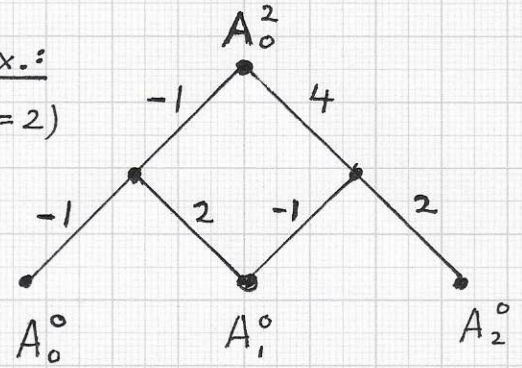
• Relationship to computation of A_0^N:

→ Leaf nodes are $A_0^0, A_1^0, \ldots, A_N^0$ (left to right).

→ Root node is A_0^N.

→ Edges in Pascal's triangle have "weights" $-(2^0), 2^1, \ldots, 2^N$.

• Ex.:
(N=2)



• A_0^N is a linear combination of $A_0^0, \ldots, A_N^0$. A_i^0 has as its coefficient the SUM OF EDGE WEIGHT PRODUCTS, considering ALL $\binom{N}{i}$ PATHS between A_i^0 and A_0^N.

RICHARDSON EXTRAPOLATION – Cont'd.

2) ... A_0^N = "**formula**"

- <u>Ex.</u>: Computation of coefficients in linear combination of A_0^2

	leaf	moves	weight products	sum
1 path	A_0^0	R, R	$(-1)\cdot(-1) = \underline{1}$	$\underline{1}$
2 paths	A_1^0	L, R	$2\cdot(-1) = \underline{-2}$	$\underline{-6}$
	A_1^0	R, L	$(-1)\cdot 4 = \underline{-4}$	
1 path	A_2^0	L, L	$2\cdot 4 = \underline{8}$	$\underline{8}$

R = right } move upwards
L = left

$$\underline{\underline{A_0^2 = 8A_2^0 - 6A_1^0 + 1A_0^0}}$$

- Computation of A_0^N in the general case:
  - → A path from a leaf to the root is encoded via a bit-string, where '0' represents a RIGHT and '1' a LEFT move upwards.
  - → A path from leaf A_i^0 to root A_0^N has an N-bit code, $b_1 b_2 \ldots b_N$, $b_i \in \{0,1\}$.
  - → The bit '0' always represents the weight -1. The bit '1' represents a power-of-two, defined by its position: 2^N in position b_N, ..., 2^1 in position b_1.

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■ RICHARDSON EXTRAPOLATION - Cont'd.

2) ... A_0^N = "formula"

➔ The bit-string of one path from leaf A_i^0 to A_0^N represents the 'weightproduct' for this path.

➔ The final coefficient of A_i^0 in the linear combination for A_0^N is the sum of all 'weight products' of all the paths from A_i^0 to A_0^N.

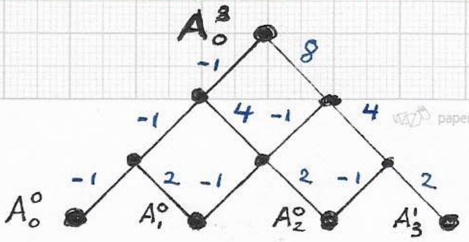
• Computations for $\underline{N=3}$:

	code	A_0^0	A_1^0	A_2^0	A_3^0
0	000	$(-1)\cdot(-1)\cdot(-1)=\underline{-1}$			
1	001		$(-1)\cdot(-1)\cdot 8=\underline{8}$		
1	010		$(-1)\cdot 4\cdot(-1)=\underline{4}$		
2	011			$(-1)\cdot 4\cdot 8=\underline{-32}$	
1	100		$2\cdot(-1)\cdot(-1)=\underline{2}$		
2	101			$2\cdot(-1)\cdot 8=\underline{-16}$	
2	110			$2\cdot 4\cdot(-1)=\underline{-8}$	
3	111	Σ	Σ	Σ	$2\cdot 4\cdot 8=64$ Σ
		−1	**14**	**−56**	**64**

• Table of all 'weight products' of all paths from A_i^0 to A_0^3.

$$\Rightarrow \underline{A_0^3 = -1A_0^0 + 14A_1^0 - 56A_2^0 + 64A_3^0}$$

Number of '1' in code defines association of 'weight products' with A_0^0, A_1^0, A_2^0 or A_3^0.



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■ RICHARDSON EXTRAPOLATION – Cont'd.

2) ... A_0^N = "**formula**"

- **Ex.:** Using a "compact encoding," compute A_0^4; consider the computation of the (absolute value of) the) coefficient for the A_2^0-term:

→ There are $\binom{4}{2} = 6$ paths from leaf A_2^0 to root A_0^2.

→ The six (6) "weight products" of these paths are defined by 6 bit strings, each one defining a path and each string containing two (2) '1' symbols.

→ The 6 4-bit strings are: _ _ 1 1, _ 1 _ 1, _ 1 1 _, 1 _ _ 1, 1 _ 1 _, 1 1 _ _ .

→ Each position in a bit string corresponds to power-of-2, WITH THE LEFT (RIGHT) BIT BEING THE LEAST (MOST) SIGNIFICANT BIT: $2^1, 2^2, 2^3, 2^4$ is the sequence of powers-of-2.

→ The notation '_' corresponds to zero (0), and the symbol '1' in a specific bit string position represents the power-of-2 value for this position.

→ Path edges have weights (-1), 2^1, 2^2, 2^3 or 2^4; **the "weight product" of a path is thus the product of path edge weights encoded by a bit string that defines the specific product of powers-of-2.** Example: _ 1 _ 1 represents the product $2^2 \cdot 2^4$.

$2^1 \quad 2^2 \quad 2^3 \quad 2^4$

(There is no need to encode -2^0 in a 5^{th} bit.)

▪ RICHARDSON EXTRAPOLATION - Cont'd.

2) ... A_0^N = **"formula"**

→ The 6 "weight products" for the 6 paths from A_2^0 to A_0^4 are:

<table>
<tr><th>2^1 2^2 2^3 2^4</th><th>product</th><th>(product's) logarithm</th><th>decimal value</th></tr>
<tr><td>_ _ | |</td><td>$2^3 \cdot 2^4 = 2^7$</td><td>$3+4=7$</td><td>128</td></tr>
<tr><td>_ | _ |</td><td>$2^2 \cdot 2^4 = 2^6$</td><td>$2+4=6$</td><td>64</td></tr>
<tr><td>_ | | _</td><td>$2^2 \cdot 2^3 = 2^5$</td><td>$2+3=5$</td><td>32</td></tr>
<tr><td>| _ _ |</td><td>$2^1 \cdot 2^4 = 2^5$</td><td>$1+4=5$</td><td>32</td></tr>
<tr><td>| _ | _</td><td>$2^1 \cdot 2^3 = 2^4$</td><td>$1+3=4$</td><td>16</td></tr>
<tr><td>| | _ _</td><td>$2^1 \cdot 2^2 = 2^3$</td><td>$1+2=3$</td><td>8</td></tr>
</table>

$\Sigma = 280$

- Computation of coefficient (absolute value) for A_2^0-term in linear combination for A_0^4: |coefficient| = 280.

<table>
<tr><th>2^1 2^2 2^3 2^4</th><th>A_1^0</th><th>A_2^0</th><th>A_3^0</th><th>A_4^0</th></tr>
<tr><td>_ _ _ _</td><td></td><td></td><td></td><td></td></tr>
<tr><td>_ _ _ |</td><td>4</td><td></td><td></td><td></td></tr>
<tr><td>_ _ | _</td><td>3</td><td></td><td></td><td></td></tr>
<tr><td>_ _ | |</td><td></td><td>7</td><td></td><td></td></tr>
<tr><td>_ | _ _</td><td>2</td><td></td><td></td><td></td></tr>
<tr><td>_ | _ |</td><td></td><td>6</td><td></td><td></td></tr>
<tr><td>_ | | _</td><td></td><td>5</td><td></td><td></td></tr>
<tr><td>_ | | |</td><td></td><td></td><td>9</td><td></td></tr>
<tr><td>| _ _ _</td><td>1</td><td></td><td></td><td></td></tr>
<tr><td>| _ _ |</td><td></td><td>5</td><td></td><td></td></tr>
<tr><td>| _ | _</td><td></td><td>4</td><td></td><td></td></tr>
<tr><td>| _ | |</td><td></td><td></td><td>8</td><td></td></tr>
<tr><td>| | _ _</td><td></td><td>3</td><td></td><td></td></tr>
<tr><td>| | _ |</td><td></td><td></td><td>7</td><td></td></tr>
<tr><td>| | | _</td><td></td><td></td><td>6</td><td></td></tr>
<tr><td>| | | |</td><td></td><td></td><td></td><td>10</td></tr>
<tr><td>Decimal Σ:</td><td>30</td><td>280</td><td>960</td><td>1024</td></tr>
<tr><td>Sign:</td><td>=</td><td>+</td><td>−</td><td>+</td></tr>
</table>

← alternating sign ←

- **Coeff. for A_0^0 = +1 (or −1)**
  - Table of logarithm values of all "weight products."
  - The sum (Σ) of the "weight products" defines the coefficients (absolute value) of A_i^0-terms:

$$A_0^4 = 1024A_4^0 - 960A_3^0 + 280A_2^0 - 30A_1^0 + 1A_0^0$$

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