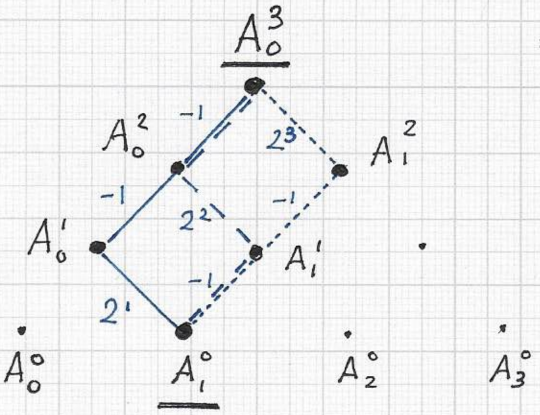
Stratovan

## ■ RICHARDSON EXTRAPOLATION – Cont'd.

2) ... $A_0^N =$ "**formula**"

→ The coefficient (absolute value) of the $A_i^0$-term in the linear combination defining $A_0^N$ is given by the SUM of all "weight products" associated with the possible paths from "node $A_i^0$" to "node $A_0^N$". Illustration for $N=3$:




⇒ Three paths from $A_1^0$ to $A_0^3$: The three "weight products" of paths ——, - - -, ······· are

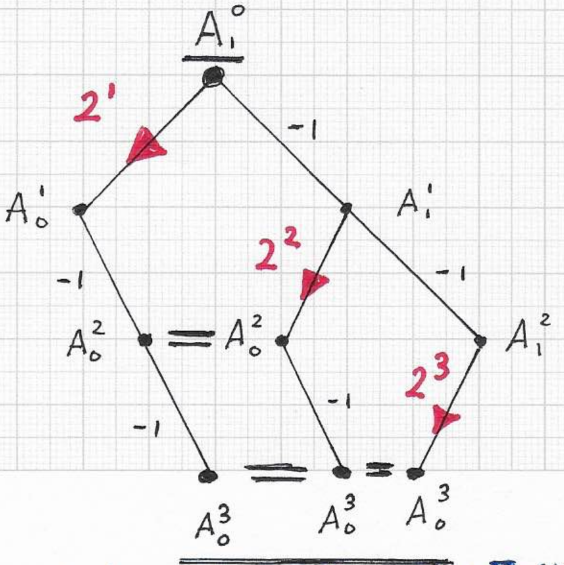
$(2^1)\cdot(-1)\cdot(-1) = 2$,

$(-1)\cdot(2^2)\cdot(-1) = 4$,

$(-1)\cdot(-1)(\cdot 2^3) = 8$.

The sum is $\underline{\underline{2+4+8=14}}$

→ A binary SEARCH TREE can be used to construct all paths from a node $A_i^0$ to $A_0^N$ – e.g., $N=3$, $i=1$:




⇒ The search tree defines the three paths from $A_1^0$ to $A_0^3$ (moving left or right at nodes):

$A_1^0 \to A_0^1 \to A_0^2 \to A_0^3$,

$A_1^0 \to A_1^1 \to A_0^2 \to A_0^3$,

$A_1^0 \to A_1^1 \to A_1^2 \to A_0^3$.

• Weight products: $\underline{\underline{2,\ 4,\ 8}} \Rightarrow \underline{\underline{\Sigma = 14}}$

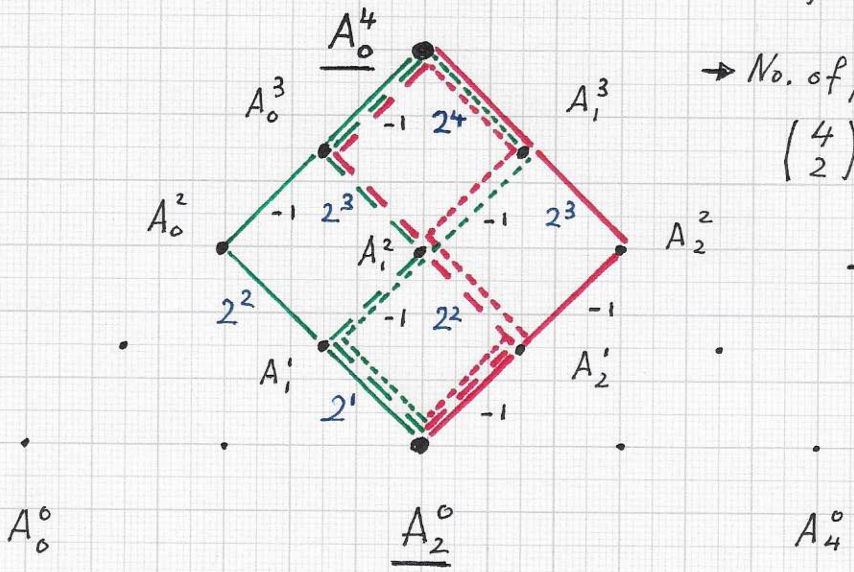
• The "LEFT MOVES" define the relevant edge weights $2^1$, $2^2$ and $2^3$.

Stratovan

## ■ RICHARDSON EXTRAPOLATION - Cont'd.

2) ... $A_0^N$ = "**formula**"

→ COEFFICIENTS (absolute value) are SUMS of PRODUCTS of edge WEIGHTS; additional ex. for $N=4$:




→ No. of paths from $A_2^0$ to $A_0^4$:

$$\binom{4}{2} = \frac{4!}{2!\,2!} = 6$$

→ Search tree generates the 6 paths, shown with solid, broken or dotted lines

⇒ • The "LEFT MOVES" define the relevant edge weights, $2^1, 2^2, 2^3, 2^4$.

• The resulting 6 path "weight products" are

$$2^1 \cdot 2^2 \cdot (-1) \cdot (-1) = 2^3,$$
$$2^1 \cdot (-1) \cdot 2^3 \cdot (-1) = 2^4,$$
$$2^1 \cdot (-1) \cdot (-1) \cdot 2^4 = 2^5,$$
$$(-1) \cdot 2^2 \cdot 2^3 \cdot (-1) = 2^5,$$
$$(-1) \cdot 2^2 \cdot (-1) \cdot 2^4 = 2^6,$$
$$(-1) \cdot (-1) \cdot 2^3 \cdot 2^4 = 2^7$$

• The resulting sum of all weight products is

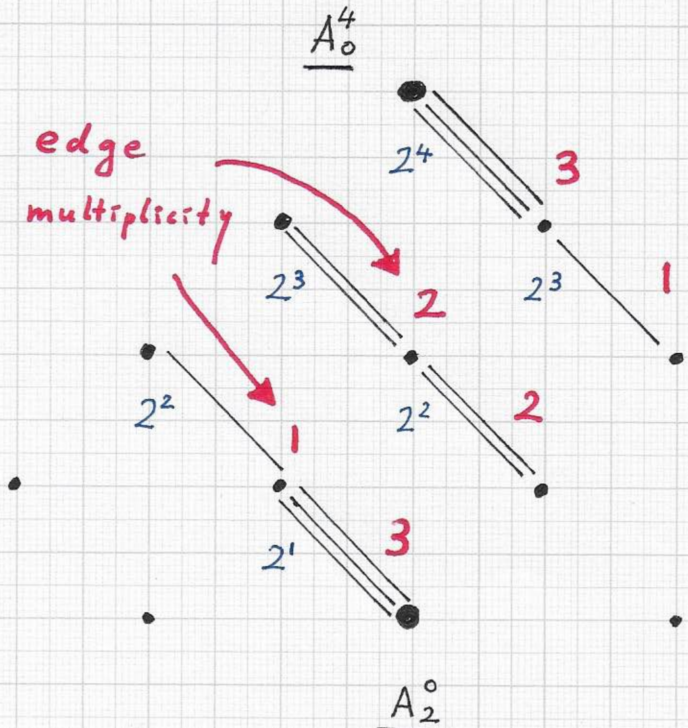
$$2^3 \cdot (1+2+4+4+8+16) = 8 \cdot 35 = \mathbf{280}.$$

~ BH

## ■ RICHARDSON EXTRAPOLATION – Cont'd.

2)... $A_0^N$ = "**formula**"

→ [Last example: $N=4$, $i=2$] Only **LEFT MOVES** are relevant, corresponding to powers-of-2; further, the left moves have associated **EDGE MULTIPLICITIES**, given by the number of paths they belong to. This data defines the needed sums of weight products.




⇒ Data needed for computation of coefficient of $A_0^4$:

(i) powers-of-2: $2^1, 2^2, 2^3, 2^4$

(ii) edge multiplicities: 3, 2, 1

⇒ Coefficient of $A_0^4$:

$$\boxed{\begin{array}{l} 2^1 \cdot 2^2 \\ + 2^1 \cdot 2^3 \\ + 2^1 \cdot 2^4 \\ + 2^2 \cdot 2^3 \\ + 2^2 \cdot 2^4 \\ + 2^3 \cdot 2^4 \end{array}} \begin{array}{l} \left.\begin{array}{l} \\ \\ \\ \end{array}\right] 3 \\ \left.\begin{array}{l} \\ \\ \end{array}\right] 2 \\ \left.\begin{array}{l} \\ \end{array}\right] 1 \end{array} \text{ TERMS}$$

$= 8+16+32+32+64+128$

$= \underline{\underline{\mathbf{280}}}$

OR:

$$\boxed{\begin{array}{l} 2^1 \cdot (2^2+2^3+2^4) \\ + 2^2 \cdot (2^3+2^4) \\ + 2^3 \cdot (2^4) \end{array}}$$

$= \underline{\underline{280}}$

Stratovan

## ■ RICHARDSON EXTRAPOLATION – Cont'd.

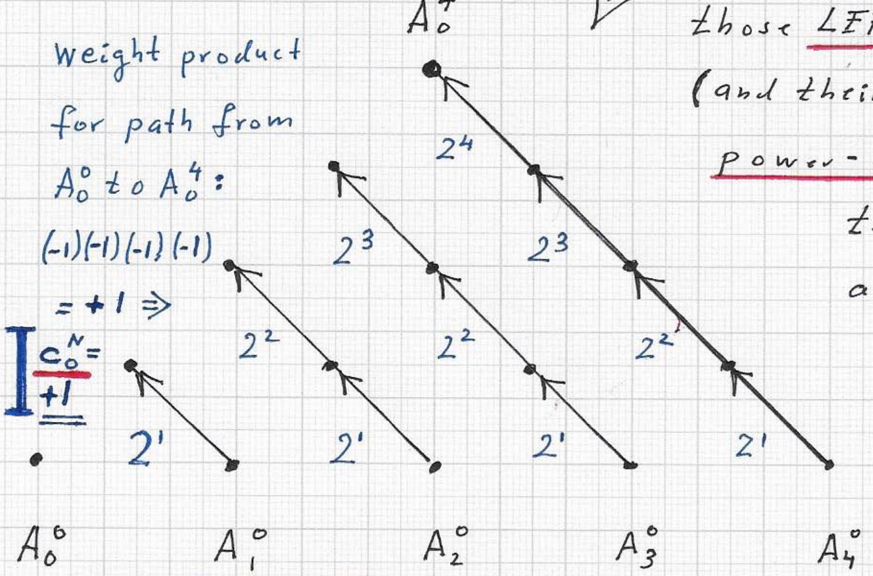
2) ... $A_0^N =$ "**formula**"

➔ ... The numerator of the expression for $A_0^N$, i.e., $(c_0^N, c_1^N, \ldots, c_N^N) \cdot (A_0^0, A_1^0, \ldots, A_N^0)^T$, is a sum-of-products-of-powers-of-2 linear combination: the coefficients $c_i^N$ are sums of products of $2^0, 2^1, 2^2, \ldots, 2^N$. This combinatorial computation can be illustrated and systematically written.

• Ex.: $N=4$

The figure shows only those LEFT-MOVE EDGES (and their associated power-of-2 weights) that are part of all the possible path from "leaves" $A_i^0$ to "root" $A_0^4$.

weight product for path from $A_0^0$ to $A_0^4$: $(-1)(-1)(-1)(-1) = +1 \Rightarrow$ $c_0^N = +1$




| | | | |
|---|---|---|---|
| $c_1^N = -1(2^1 + 2^2 + 2^3 + 2^4) = -30$ | $c_2^N = +1(2^1 2^2 + 2^1 2^3 + 2^1 2^4 + 2^2 2^3 + 2^2 2^4 + 2^3 2^4) = +280$ | $c_3^N = -1(2^1 2^2 2^3 + 2^1 2^2 2^4 + 2^1 2^3 2^4 + 2^2 2^3 2^4) = -960$ | $c_4^N = +1(2^1 2^2 2^3 2^4) = +1024$ |

**no. of products:** $\binom{4}{1}$ $\binom{4}{2}$ $\binom{4}{3}$ $\binom{4}{4}$

■ RICHARDSON EXTRAPOLATION – Cont'd.

2) ... $A_0^N$ = "**formula**" [see previous page!]

• Ex.: $N=4$ $\quad A_0^4 = \left( + \underline{1}\, A_0^0 - \underline{30}\, A_1^0 + \underline{280}\, A_2^0 - \underline{960}\, A_3^0 + \underline{1024}\, A_4^0 \right) / D_4.$

➔ Table of data for coefficient computation:

$^{*}\underline{S(i)} = \begin{cases} +, & (N-i) \text{ even} \\ -, & (N-i) \text{ odd} \end{cases}$,

| $i$ | $\binom{4}{i}$ | powers-of-2 combinations: $2^1$ | $2^2$ | $2^3$ | $2^4$ | products | sum | $S^{*}$ |
|---|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | 1 | $\underline{1}$ | + |
| 1 | 4 | • | | | | $2^1 = 2$ | | |
| | | | • | | | $2^2 = 4$ | | |
| | | | | • | | $2^3 = 8$ | | |
| | | | | | • | $2^4 = 16$ | $\underline{30}$ | − |
| 2 | 6 | • | • | | | $2^1 2^2 = 8$ | | |
| | | • | | • | | $2^1 2^3 = 16$ | | |
| | | • | | | • | $2^1 2^4 = 32$ | | |
| | | | • | • | | $2^2 2^3 = 32$ | | |
| | | | • | | • | $2^2 2^4 = 64$ | | |
| | | | | • | • | $2^3 2^4 = 128$ | $\underline{280}$ | + |
| 3 | 4 | • | • | • | | $2^1 2^2 2^3 = 64$ | | |
| | | • | • | | • | $2^1 2^2 2^4 = 128$ | | |
| | | • | | • | • | $2^1 2^3 2^4 = 256$ | | |
| | | | • | • | • | $2^2 2^3 2^4 = 512$ | $\underline{960}$ | − |
| 4 | 1 | • | • | • | • | $2^1 2^2 2^3 2^4 = 1024$ | $\underline{1024}$ | + |

$$\Rightarrow \text{sum}_i \;(i=1,\dots,N) = \sum_{\binom{N}{i} \text{ product combinations}} \prod 2^{e_1} \cdot \ldots \cdot 2^{e_i} \; : \text{mutually different exponents } e_1,\dots,e_i \in \{1,\dots,N\}$$

$$\boxed{\text{sum}_0 = 1}.$$

~BH

Stratovan

## • RICHARDSON EXTRAPOLATION – Cont'd.

2) … $A_0^N$ = "formula"

**• Ex.: $N = 5$** – "Combinatorial structure," coefficient computation

| $i$ | $2^1$ | $2^2$ | $2^3$ | $2^4$ | $2^5$ | coefficient |
|---|---|---|---|---|---|---|
| 0 | | | | | | −1 |
| 1 | • | | | | | |
| | | • | | | | |
| | | | • | | | |
| | | | | • | | |
| | | | | | • | +62 |
| 2 | • | • | | | | |
| | • | | • | | | |
| | • | | | • | | |
| | • | | | | • | |
| | | • | • | | | |
| | | • | | • | | |
| | | • | | | • | |
| | | | • | • | | |
| | | | • | | • | |
| | | | | • | • | −1240 |
| 3 | • | • | • | | | |
| | • | • | | • | | |
| | • | | • | • | | |
| | • | | • | | • | |
| | • | | | • | • | |
| | | • | • | • | | |
| | | • | • | | • | |
| | | • | | • | • | |
| | • | • | | | • | |
| | | | • | • | • | +9920 |
| 4 | • | • | • | • | | |
| | • | • | • | | • | |
| | • | • | | • | • | |
| | • | | • | • | • | |
| | | • | • | • | • | −31744 |
| 5 | • | • | • | • | • | +32768 |

### • FORMULA FOR $A_0^N$:

$$A_0^N = \frac{\sum_{i=0}^{N} c_i^N A_i^0}{D_N}, \quad N \geq 1,$$

where:

→ $D_N = \prod_{i=1}^{N} (2^i - 1)$

→ $c_i^N = s(i) \cdot sum_i$,

where

→ $s(i) = \begin{cases} +, & (N-i) \text{ even} \\ -, & (N-i) \text{ odd} \end{cases}$

→ $sum_i = \sum 2^{e_1} \cdot \ldots \cdot 2^{e_i}, \; i = 1 \ldots N,$

$\binom{N}{i}$ power-of-2 products with $i$ mutually different exponents $e_1, \ldots, e_i \in \{1, \ldots, N\}$

and

→ $sum_0 = 1$.

~ BH