■ RICHARDSON EXTRAPOLATION – Cont'd.

2) ... A_0^N = "**formula**"

- Towards a formula that is "more compact" (and computationally less expensive) than the one given on previous page! What is the value of $\sum_i 2^{e_1} \cdot \ldots \cdot 2^{e_i}$, $i = 1 \ldots N$? Closed-form formula?

- Consider example $N=6$

$$A_0^6 = \big(+1\, A_0^0 - 126\, A_1^0 + 5208\, A_2^0 - 89280\, A_3^0 + 666624\, A_4^0 - 2064384\, A_5^0 + 2097152\, A_6^0 \big) / D_6 ,$$

where

$$D_6 = \prod_{i=1}^{6} (2^i - 1) = 1 \cdot 3 \cdot 7 \cdot 15 \cdot 31 \cdot 63 = 615\,195$$

	i	2^1	2^2	2^3	2^4	2^5	2^6	products*	sums
1	0								1
	1	•						2	
			•					4	
6				•				8	
					•			16	
						•		32	
							•	64	126
	2	•	•					8	
		•		•				16	
		•			•			32	
		•				•		64	
		•					•	128	
			•	•				32	
			•		•			64	
15			•			•		128	
			•				•	256	
				•	•			128	
				•		•		256	
				•			•	512	
					•	•		512	
					•		•	1024	
						•	•	2048	5208

↑

$\binom{6}{i}_{i=0\ldots6} = (1, 6, 15, 20, 15, 6, 1)$

* products = products of powers-of-2, indicated by '•'

■ RICHARDSON EXTRAPOLATION – Cont'd.

2)... A_0^N = "formula"

	i	2^1	2^2	2^3	2^4	2^5	2^6	products	sum
	3	•	•	•				64	
		•	•		•			128	
		•	•			•		256	
		•	•				•	512	
		•		•	•			256	
		•		•		•		512	
		•		•			•	1024	
		•			•	•		1024	
		•			•		•	2048	
20		•				•	•	4096	
			•	•	•			512	
			•	•		•		1024	
			•	•			•	2048	
			•		•	•		2048	
			•		•		•	4096	
			•			•	•	8192	
				•	•	•		4096	
				•	•		•	8192	
				•		•	•	16384	
					•	•	•	32768	89280
	4	•	•	•	•			1024	
		•	•	•		•		2048	
		•	•	•			•	4096	
		•	•		•	•		4096	
		•	•		•		•	8192	
		•	•			•	•	16384	
15		•		•	•	•		8192	
		•		•	•		•	16384	
		•		•		•	•	32768	
		•			•	•	•	65536	
			•	•	•	•		16384	
			•	•	•		•	32768	
			•	•		•	•	65536	
			•		•	•	•	131072	
				•	•	•	•	262144	666624
	5	•	•	•	•	•		32768	
		•	•	•	•		•	65536	
6		•	•	•		•	•	131072	
		•	•		•	•	•	262144	
		•		•	•	•	•	524288	
			•	•	•	•	•	1048576	2064384
1	6	•	•	•	•	•	•	2097152	2097152

■ RICHARDSON EXTRAPOLATION – Cont'd.

2) ... $A_0^N =$ "**formula**"

- Can one 'deduce' a generating function for the "sums-of-products-of-powers-of-2"?

→ Table of (absolute) values of coefficients c_i^N, $N = 0 \ldots 6$:

N \ i	0	1	2	3	4	5	6
0	$1 = 2^0$						
1	1	$2 = 2^1$					
2	1	6	$8 = 2^3$				
3	1	14	56	$64 = 2^6$			
4	1	30	280	960	1624 $= 2^{10}$		
5	1	62	1240	9920	31744	32768 $= 2^{15}$	
6	1	126	5208	89280	666624	2064384	2097152 $= 2^{21}$

→ Values of D_i, $i = 0 \ldots 6$: 1, 1, 3, 21, 315, 9765, 615195

● **INSIGHTS:** → $|c_N^0| = 1$ (column 0)

→ $|c_N^N| = 2^0 \cdot 2^1 \cdot 2^2 \cdot 2^3 \cdot \ldots \cdot 2^N$

$= 2^{0+1+2+3+\ldots+N}$

$= 2^{\sum_{i=0}^N i}$

$= 2^{N \cdot (N+1)/2}$ (diagonal)

→ $|c_N^1| = 2^1 + 2^2 + 2^3 + \ldots + 2^N$

$= \sum_{i=1}^N 2^i \overset{✪}{=} 2(2^N - 1)$ (column 1)

✪ finite geometrical sum:

$$\sum_{i=1}^N 2^i = \frac{2^{N+1} - 1}{2 - 1} - 2^0 = 2^{N+1} - 2 = 2(2^N - 1)$$

Stratovan

■ RICHARDSON EXTRAPOLATION – Cont'd.

2) ... A_0^N = "formula"

• **INSIGHTS:** $\Rightarrow$ $|c_N^2| = |c_{N-1}^2| + \sum_{i=N+1}^{2N-1} 2^i$ (column 2)

$$\left(\begin{aligned} |c_2^2| &= 2^3 = 8 \\ |c_3^2| &= 8 + 2^4 + 2^5 = 56 \\ |c_4^2| &= 56 + 2^5 + 2^6 + 2^7 = 280 \\ |c_5^2| &= 280 + 2^6 + 2^7 + 2^8 + 2^9 = 1240 \\ |c_6^2| &= 1240 + 2^7 + 2^8 + 2^9 + 2^{10} + 2^{11} = 5208 \end{aligned} \right)$$

$$\Downarrow$$

$$\underline{|c_N^2|} = \sum_{i=2}^{N} \sum_{j=i+1}^{2i-1} 2^j = \underline{\frac{4}{3}(2^N-2)(2^N-1)}$$

$\Rightarrow |c_N^3| = ?$

$$\left(\begin{aligned} |c_3^3| &= 2^6 = 64 \\ |c_4^3| &= 2^6 \\ &\quad + 2^7 + 2^8 + 2^9 = 960 \\ |c_5^3| &= 2^6 \\ &\quad + 2^7 + 2^8 + 2^9 \\ &\quad + 2^8 + 2^9 + 2 \cdot 2^{10} + 2^{11} + 2^{12} = 9920 \\ |c_6^3| &= 2^6 \\ &\quad + 2^7 + 2^8 + 2^9 \\ &\quad + 2^8 + 2^9 + 2 \cdot 2^{10} + 2^{11} + 2^{12} \\ &\quad + 2^9 + 2^{10} + 2 \cdot 2^{11} + 2 \cdot 2^{12} + 2 \cdot 2^{13} + 2^{14} + 2^{15} = 89280 \end{aligned} \right)$$

$$\Rightarrow \underline{|c_N^3| = \sum_{i=6}^{N+3} \sum_{j=i}^{3i-12} 2^j + \sum_{i=6}^{N+3} \sum_{j=i+2}^{3i-14} 2^j = \ldots}$$ (at least for $N = 3 \ldots 6$)

RICHARDSON EXTRAPOLATION – Cont'd.

2) … A_0^N = "**formula**"

(Note: $|c_6^3| = 2^6$
$+ 2^7 + 2^8 + 2^9$
$+ 2^8 + 2^9 + 2^{10} + 2^{11} + 2^{12}$
$+ 2^9 + 2^{10} + 2^{11} + 2^{12} + 2^{13} + 2^{14} + 2^{15}$ $\Big\} = \sum_{i=6}^{N+3} \sum_{j=i}^{3i-12} 2^j$

$+ 2^{10} + 2^{11} + 2^{12} + 2^{13}$ $\Big\} + \sum_{i=6}^{N+3} \sum_{j=i+2}^{3i-14} 2^j$,

$N = 6$)

→ $|c_N^4| = ?$

($|c_4^4| = 2^{10} = 1024$

$|c_5^4| = 2^{10}$
$+ 2^{11} + 2^{12} + 2^{13} + 2^{14} = 31744$

$|c_6^4| = 1 \cdot 2^{10} + 3 \cdot 2^{11} + 7 \cdot 2^{12} + 15 \cdot 2^{13} + 31 \cdot 2^{14} = 666624$

$\left(= \sum_{i=0}^{4} (2^{i+1} - 1) \cdot 2^{i+10} = \frac{1024}{3} (2^{4+1} - 1)(2^{4+2} - 1) \right)$)

→ $|c_N^5| = ?$

($|c_5^5| = 2^{15} = 32768$

$|c_6^5| = 2^{21} - 2^{15} = 2064384$)

→ $\underline{|c_N^6|} \underset{N=6}{=} |c_6^6| = 2^{21} = 2^{0+1+2+\ldots+6}$

$\underline{= 2^{\sum_{i=0}^{6} i} = 2^{\frac{6}{2} \cdot (6+1)}}$

- **References for GENERATING FUNCTIONS:**

1) Donald E. KNUTH, The Art of Computer Programming, Vol. 1, pp. 82–84, 87–93.

2) Herbert S. WILF, generatingfunctionology, Academic Press.