■ RICHARDSON EXTRAPOLATION - Cont'd.

2)... A_0^N = "formula" *FINAL*

- The standard definition of Richardson extrapolation uses a RECURSIVE formula computing a new extrapolation value from already computed values:

$$A_i^j = \frac{2^j A_{i+1}^{j-1} - A_i^{j-1}}{2^j - 1}$$

(see NOTES dated 1/29/2021)

- The goal is to compute the value A_0^N DIRECTLY from given values $A_0^0, \ldots, A_N^0$.

- For efficiency one should compute A_0^N as $(N \geq 1)$

$$A_0^N = \frac{\sum_{i=0}^N c_i^N A_i^0}{D_N},$$

where:

→ $D_N = \prod_{i=1}^N (2^i - 1)$,

→ $\text{sign}(N,i) = s(N,i) = \begin{cases} + , & (N-i) \text{ even} \\ - , & (N-i) \text{ odd} \end{cases}$,

→ $c_i^N = s(N,i) \cdot |c_i^N|$

(ultimate goal: $\lim_{N \to \infty} A_0^N = ?$)

- The needed coefficients' absolute values are

*
*
*

$$\boxed{|c_i^N| = \frac{2^i}{\prod_{j=1}^i (2^j - 1)} \cdot \prod_{j=0}^{i-1} (2^N - 2^j)},$$

$i = 1 \ldots N$ and $|c_0^N| = 1$.

(symbolic computation: www.wolframalpha.com)

* FORMULA FOR RICHARDSON EXTRAPOLATION - NOT USING RECURSION!!! *

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■ RICHARDSON EXTRAPOLATION – Cont'd.

2) $A_0^N = \left(\sum_{i=0}^{N} c_i^N A_i^0 \right) / D_N$

- Ex.: Application of the "coefficient formula" for $|c_i^N|$

- Table of coefficients:

	$i=0$	1	2	3	4	5	6
$N=0$	**1** – 2^0						
1	**1**	**2** – 2^1					
2	**1**	6	**8** – 2^3				
3	**1**	14	56	**64** – 2^6			
4	**1**	30	280	960	**1024** – 2^{10}		
5	**1**	62	1240	9920	31744	**32768** – 2^{15}	
6	**1**	126	5208	89280	666624	2064384	**2097152** – 2^{21}
	1	…					
	⋮						

$|c_N^N| = 2^{N(N+1)2}$

$|c_0^N| = 1$

$\rightarrow |c_1^N| = \frac{2^1}{1} \cdot \prod_{j=0}^{0} (2^N - 2^j) = 2 \cdot (2^N - 1)$

$\rightarrow |c_2^N| = \frac{2^2}{1 \cdot 3} \cdot \prod_{j=0}^{1} (2^N - 2^j) = \frac{4}{3} (2^N - 1)(2^N - 2)$

$\rightarrow |c_3^N| = \frac{2^3}{1 \cdot 3 \cdot 7} \cdot \prod_{j=0}^{2} (2^N - 2^j) = \frac{8}{21} (2^N - 1)(2^N - 2)(2^N - 4)$

$\rightarrow |c_4^N| = \frac{2^4}{1 \cdot 3 \cdot 7 \cdot 15} \cdot \prod_{j=0}^{3} (2^N - 2^j) = \frac{16}{315} (2^N - 1)(2^N - 2)(2^N - 4)(2^N - 8)$

$\rightarrow |c_5^N| = \frac{2^5}{1 \cdot 3 \cdot 7 \cdot 15 \cdot 31} \cdot \prod_{j=0}^{4} (2^N - 2^j) = \frac{32}{9765} (2^N - 1)(2^N - 2)(2^N - 4)(2^N - 8)(2^N - 16)$

$\rightarrow |c_6^N| = \frac{2^6}{1 \cdot 3 \cdot \ldots \cdot 63} \cdot \prod_{j=0}^{5} (2^N - 2^j) = \frac{64}{615195} (2^N - 1)(2^N - 2) \ldots (2^N - 32)$

■ RICHARDSON EXTRAPOLATION – Cont'd.

2) … A_0^N: NOTE concerning COMPLEMENTARY computation of $|c_i^N|$ values

• Ex.: $N = 4$, $i = 0 \ldots 4$; complementary products of powers-of-2

i	2^1	2^2	2^3	2^4	$\Pi^\bullet$	Π°	$\Sigma^\bullet$	Σ°
0	○	○	○	○		1024	1	1024
1	•	○	○	○	2	512		
	○	•	○	○	4	256		
	○	○	•	○	8	128		
	○	○	○	•	16	64	30	960
2	•	•	○	○	8	128		
	•	○	•	○	16	64		
	•	○	○	•	32	32		
	○	•	•	○	32	32		
	○	•	○	•	64	16		
	○	○	•	•	128	8	280	280
3	•	•	•	○	64	16		
	•	•	○	•	128	8		
	•	○	•	•	256	4		
	○	•	•	•	512	2	960	30
4	•	•	•	•	1024		1024	1

• power-of-2 "ON"

○ complementary power-of-2 "ON"

$\Pi^\bullet$, Π° products of powers-of-2

$\Sigma^\bullet$, Σ° resulting sums of products of powers-of-2

⇒ $\Sigma^\bullet$ and Σ° are complementary:

$$\Pi = \prod_{i=1}^{4} 2^i = 2^{10} = 1024$$

$$\Rightarrow \Pi = \Pi^\bullet \cdot \Pi^\circ$$

$$\Rightarrow \Pi^\circ = \Pi / \Pi^\bullet$$

⇒ $\Sigma^\bullet$ and Σ° are complementary:

30 complementary to 960

280 " " 280

960 " " **30**

⇒ ONLY NEED TO COMPUTE VALUES UP TO $i = N/2$ (N EVEN) OR $i = (N+1)/2$ (N ODD).

■ RICHARDSON EXTRAPOLATION – Cont'd.

2) ... A_0^N: Complementary $|c_i^N|$ computation

• Ex.: $N=4$; $A_0^4 = (+1A_0^0 - 30A_1^0 + 280A_2^0 - 960A_3^0 + 1024A_4^0)/315$

compl.

compl.

compl.

$\Rightarrow |c_i^4|$ is complementary to $|c_{4-i}^4|$, $i = 0 \dots 2$

$\Rightarrow$ Complementary computations:

$\rightarrow |c_1^4| = \Sigma^{\bullet} = 30$, $|c_3^4| = \Sigma^{\circ} = \Sigma(\Pi_{all}^{\circ}) = \Sigma(\Pi / \Pi_{all}^{\bullet})$

$$= \Pi \cdot \Sigma\left(\frac{1}{\Pi^{\bullet}}\right)_{all}$$

$$= 1024 \cdot \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}\right)$$

$$= 1024 \cdot \frac{15}{16} = \mathbf{960}$$

$\rightarrow |c_2^4| = \Sigma^{\bullet} = 280$, $|c_2^4| = \Sigma^{\circ} = \Sigma(\Pi_{all}^{\circ}) = \Sigma(\Pi / \Pi_{all}^{\bullet})$

$$= 1024 \cdot \left(\frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128}\right)$$

$$= 1024 \cdot \frac{35}{128} = \mathbf{280}$$

"$|c_2^4|$ is SELF-complementary, since $N=4$ is even."

$\Rightarrow$ Potential for increased efficiency:

- N even $\rightarrow$ compute $|c_0^N|, \dots, |c_{N/2}^N|$
- N odd $\rightarrow$ " $|c_0^N|, \dots, |c_{(N+1)/2}^N|$

COMPLEMENTARY COMPUTATION OF MISSING VALUES OF $|c_{N-j}^N|$ ☺

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