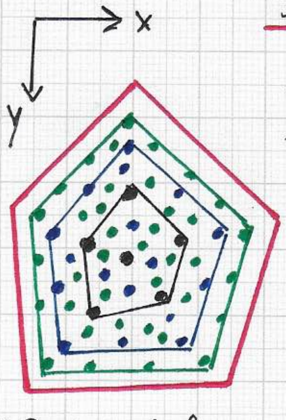
Stratovan

■ RICHARDSON EXTRAPOLATION FOR FEATURE SPACE

- **Problem:** Consider the case of a high-dimensional feature space; assume that only a 'relatively small number' of discrete samples (with associated feature points/vectors) is available to define the finite sub-region in feature space that corresponds to a specific (material) class. How could one possibly use Richardson extrapolation to generate a 'high-quality' approximation of this class-specific sub-region?

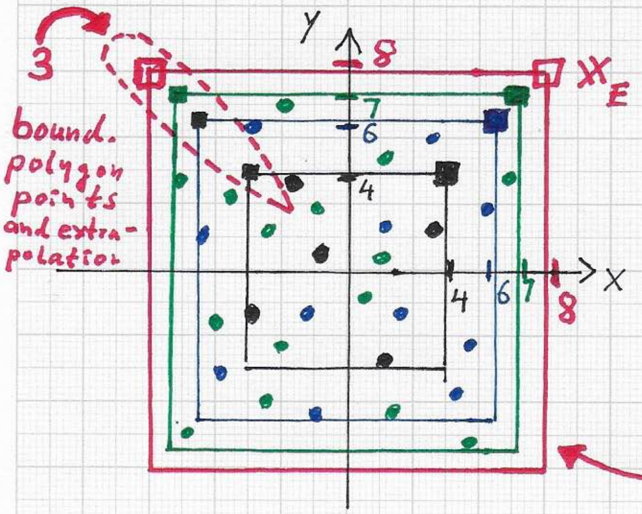
- **Solution:** [Concept of solution approach] Consider a stochastic approach; given is a 'relatively small' set of feature points/vectors; from this set select, for example, 3 sub-sets with increasing numbers of selected feature points/vectors. The union of the 3 sub-sets is the original set of feature points/vectors, and the 'density' of feature points/vectors of the 3 sub-sets, of increasing cardinality, also increases in a 'proper geometrical progression' (leading to decreasing feature point distances in their 3 sub-sets). For example, one can explain the extrapolation via convex hull extrapolation (see illustration). One can apply Richardson extrapolation to the convex hull boundary polygons and extrapolate them to the RED boundary of the class-specific region.



- Original feature point set = union of black, blue and green points. BLACK, BLUE, GREEN feature point sub-sets approximate the needed feature region in xy-space with increasing accuracy. Richardson extrapolation generates the desired region with RED boundary.

RICHARDSON EXTRAPOLATION FOR FEATURE SPACE – Cont'd.

- Ex.: [Simplified, special case] Compute convex hull boundary of 'expanded region' in feature space of feature points, by applying Richardson extrapolation to the 3 bounding squares of 3 feature point subsets:



→ Extrapolate the 3 'corresponding points' $\binom{4}{4}$, $\binom{6}{6}$, $\binom{7}{7}$:

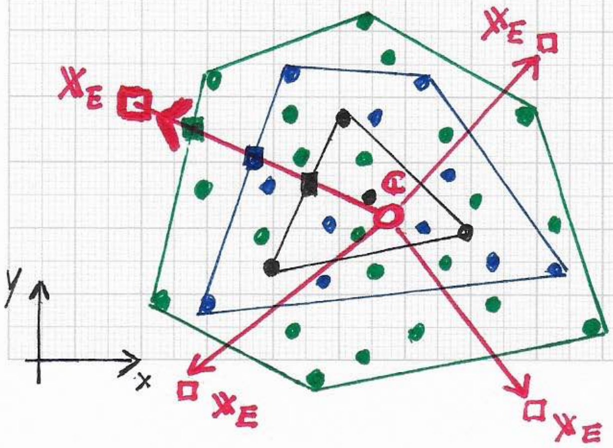
$$X_E = \left(8\binom{7}{7} - 6\binom{6}{6} + \binom{4}{4}\right)/3 = \binom{8}{8} = \square$$

Polygon = result of 'extrapolating' boundary polygons of 3 feature point sub-set boundary polygons

{•} Feature pnt. sub-set 1

{•} " " " 2

{•} " " " 3



→ More general case: 3 feature point sub-sets, {•}, {•}, {•}, with individual boundary polygons of their convex hulls;

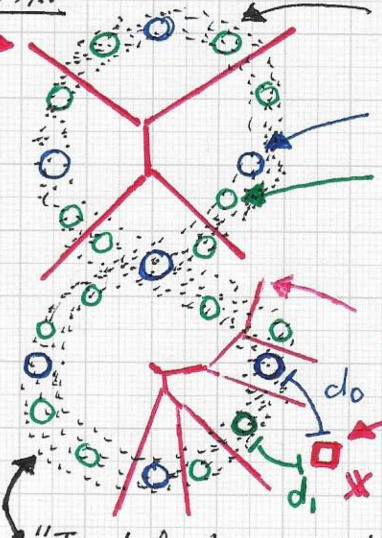
⇒ extrapolate with respect to Line C, ■, ■, ■ to compute $X_E = \square$.

(The point C is the set's centroid; points ■, ■, and ■ have associated 'spacing parameter values Δ, $\Delta/2$ and $\Delta/4$.)

■ RICHARDSON EXTRAPOLATION FOR FEATURE SPACE – Cont'd.

• **Approach:** Given a (relatively) LARGE set of feature points/vectors for a specific (material) class, use randomly chosen sub-sets to "represent" the region in HIGH-DIMENSIONAL feature space corresponding to the class. Use the concept of VORONOI DIAGRAM induced proximity to define the region, and use Richardson extrapolation to determine whether a specific (new) feature point/vector belongs to the region.

• **Ex.:**



MANY feature points in 2D space

part of Voronoi diagram of only S_0

{O} randomly selected sub-set S_0, spacing Δ_0

{O} randomly selected sub-set S_1, ADDED TO S_0, spacing Δ_1

part of Voronoi diagram of $S_0 \cup S_1$

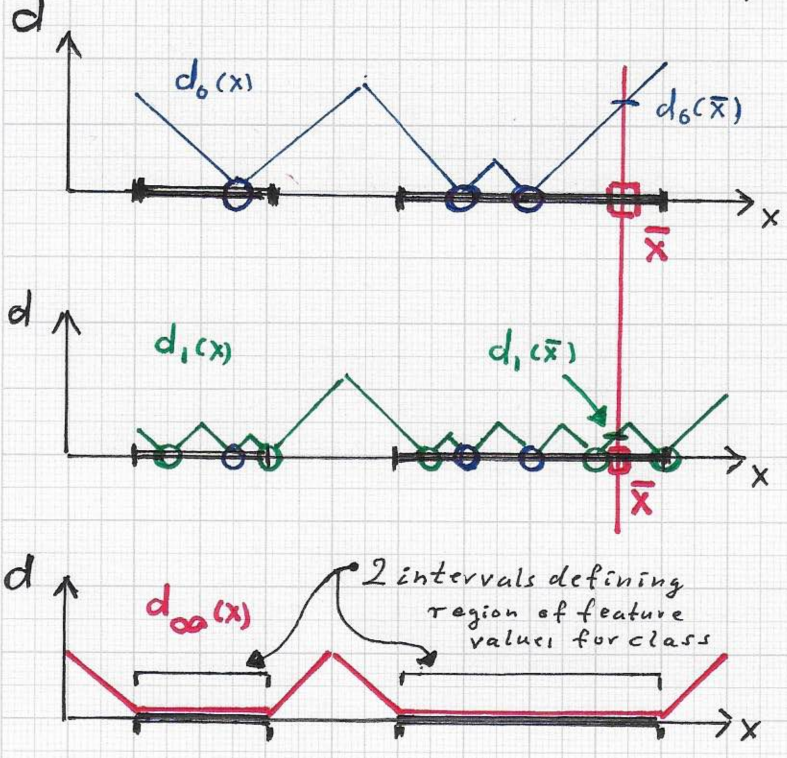
feature point ※ with distances d_0 and d_1 between ※ and (tile center) points O and O of S_0 and $(S_0 \cup S_1)$ diagrams

"Exact feature point region ⇔ infinitely dence sampling"

⇒ Compute distances for point ※ and tile centers of Voronoi diagram tiles of "geometrically decreasing spacing" between points of feature point sub-sets. Estimate the distance between ※ and the unknown exact feature point region via Richardson extrapolation.

RICHARDSON EXTRAPOLATION FOR FEATURE SPACE – Cont'd.

• **Ex.:** Consider a 1D feature space where the exact region of a specific class is the union of two intervals. (It is assumed, for simplicity, that class-specific exact feature points occur with equal probability for all x-values in these intervals. One could also assume and model more complicated value probabilities.)



→ distance function d: $d \geq 0$, non-negative distance between x and closest o

→ {o} = feature point sub-set S_0

→ {o} ∪ {o} = $S_0 \cup S_1$

→ two selected sub-sets define distance functions $d_0(x)$, $d_1(x)$

→ infinite sampling density (spacing $\Delta = 0$) leads to $d_\infty(x)$

→ **NEEDED: $d(\bar{x}) = ?$**

• For example, if one has 3 distance functions d_0, d_1, and d_2, with associated spacing values Δ_0, $\frac{\Delta_0}{2}$, and $\frac{\Delta_0}{4}$, the extrapolated distance value for $\bar{x}$ is $d_E(\bar{x}) = \left(8 d_2(\bar{x}) - 6 d_1(\bar{x}) + 1 d_0(\bar{x})\right) / 3$

• $d_E(\bar{x})$ can be negative $\Rightarrow$ $\overline{d_E}(\bar{x}) = \begin{cases} 0, & \text{if } d_E(\bar{x}) < 0 \\ d_E(\bar{x}), & \text{otherwise} \end{cases}$

• Classification: $\overline{d_E}(\bar{x}) < \varepsilon(\text{class}) \Rightarrow \bar{x}$ belongs to class

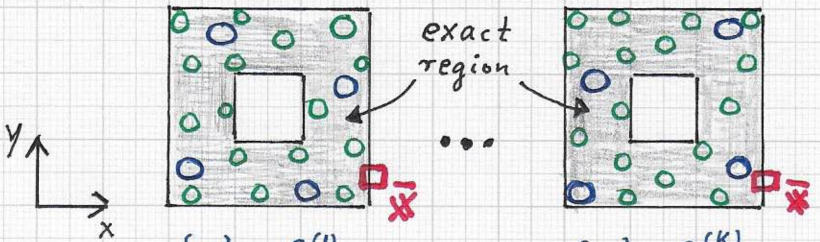
■ RICHARDSON EXTRAPOLATION FOR FEATURE SPACE - Cont'd.

- **Goal:** Achieving classification efficiency and (a high degree of) correctness by using only SMALL sub-sets from given LARGE sample/training data set as input for Richardson extrapolation

  ⇒ **Stochastic approach:** Randomly generate multiple SMALL sub-sets from given LARGE sample set to compute 'sufficiently good' Richardson extrapolation values leading to statistically 'acceptable' classification!

- **Ex.:** Randomly select sub-sets S_0, S_1, ... from large sample set and perform extrapolation for each selection

  - selection 1 ... - selection K



$\{o\} = S_0^{(1)}$ ... $\{o\} = S_0^{(K)}$

$\{o\} = S_1^{(1)}$ ... $\{o\} = S_1^{(K)}$

⋮ **STOCHASTIC OPTIMIZATION** ⋮

→ compute extra-polated values for $\overline{d}_E^{(1)}(\overline{\mathbb{X}}), \ldots, \overline{d}_E^{(K)}(\overline{\mathbb{X}})$.

→ use MINIMUM or AVERAGE of $\overline{d}_E^{(j)}(\overline{\mathbb{X}})$ values as final distance value for $\overline{\mathbb{X}}$.

- **Cost:** Computational cost is dominated by the cost of determining (quickly!) the points (in high-dimensional feature space) in $S_0^{(1)}, \ldots, S_0^{(K)}, S_1^{(1)}, \ldots, S_1^{(K)}, \ldots$ closest to $\overline{\mathbb{X}}$.

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