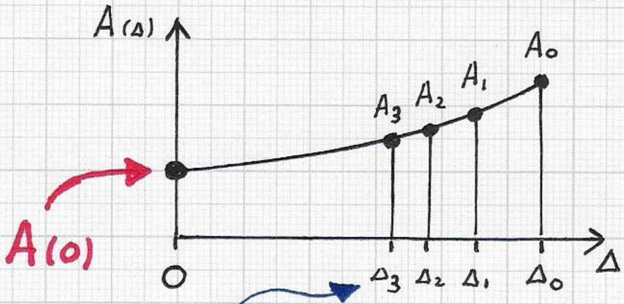
RICHARDSON EXTRAPOLATION FOR FEATURE SPACE - Cont'd.

... (vi) POLYNOMIAL 'RICHARDSON-LIKE EXTRAPOLATION'

- for cases not permitting halfing of values of a spacing parameter Δ



Δ_i-values do NOT satisfy geometric progression.

given: "data spacing values" $\Delta_0, \Delta_1, \Delta_2, \Delta_3, \ldots$ and corresponding approximations $A_0, A_1, A_2, A_3, \ldots$

wanted: $A_\infty = A(\Delta_\infty) = A(0)$

→ (Assuming that A_i-values are 'not too noisy' and define a 'nearly monotonically increasing or decreasing sequence of values',) **MODEL $A(\Delta)$ WITH A POLYNOMIAL:**

$$A(\Delta) = \sum_{i=0}^{n} c_i \Delta^i \,, \quad \text{with } A(\Delta_j) = A_j$$

There are $J+1$ spacing values $\Delta_0, \ldots, \Delta_J$ and $J+1$ approximations $A_0, \ldots, A_J$. ⇒ Solve linear system

$$\Delta_j^0 c_0 + \Delta_j^1 c_1 + \ldots + \Delta_j^n c_n = A_j \,, \quad j = 0 \ldots J$$

a) $n = J$: $c_0, \ldots, c_n$ uniquely defined

b) $n < J$: over-determined system ⇒ compute least-squares solution

(Note: $n < J$ could handle noisy data.)

■ RICHARDSON EXTRAPOLATION FOR FEATURE SPACE – Cont'd.

...(vi) → Relationship to performing extrapolation based on a relatively SPARSE FEATURE DATA SET DEFINED OVER A HIGH-DIMENSIONAL FEATURE SPACE (i.e., a region containing given feature data):

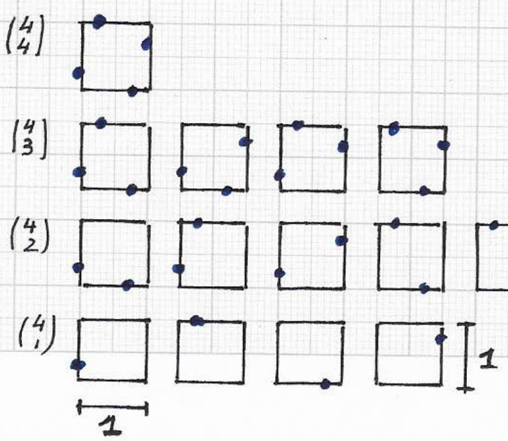
- given: **N** feature points in **d**-dimensional feature space, lying in a finite region with hyper-volume **V**

  ⇒ **The full N-point data set has associated spacing $\Delta_J = \sqrt[d]{V/N}$.**

  (I.e., the N-point data set has maximal point density and minimal spacing.)

- goal: extrapolated value of $A(0)$, based on approximation values $A_J, A_{J-1}, \ldots, A_0$, where randomly selected sub-sets of the N-point set define the A_j-values.

(vii) Computation of A_j-values using a STOCHASTIC method.



- Ex.: $N=4,\ d=2,\ V=1$

  ⇒ non-empty sub-sets have cardinality 4, 3, 2, 1

  ⇒ associated spacing: $\sqrt{1/4}, \sqrt{1/3}, \sqrt{1/2}, \sqrt{1/1}$

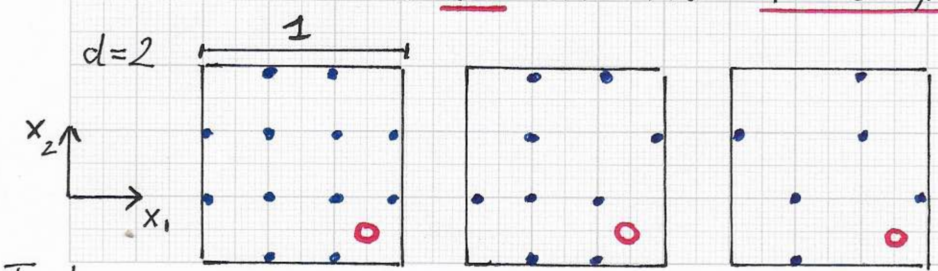
  ⇒ e.g., set $\Delta_3 = 1/2$, $\Delta_2 = \sqrt{3}/3$, $\Delta_1 = \sqrt{2}/2$, $\Delta_0 = 1$.

▪ RICHARDSON EXTRAPOLATION FOR FEATURE SPACE – Cont'd.

... (vii) STOCHASTIC method for extrapolation with feature points/vectors in HIGH-DIMENSIONAL SPACE

→ N given feature points represent a 'point cloud' in high-dim. feature space; these points are arbitrarily scattered and are gridless. Sub-sets of the N-point data set can be generated in a stochastic fashion. The origal N-point set and all generated sub-sets have associated spacing values (Δ-values).

→ In the context of CLASSIFICATION, a new sample and its associated feature point/vector is provided. It is possible to compute 'distance approximations' between the new sample's feature point and the N-point set and the stochastically generated sub-sets. The resulting distance approximations A_j can be associated with the spacing values Δ_j of the respective feature point sets. THUS, ONE CAN COMPUTE THE LIMIT VALUE $A(0)$ and use this value for sample classification.



• Feature Space

$N = 12$ | $\Delta_2 = 1/\sqrt{12} = 0.29$

$0.75N = 9$ | $\Delta_1 = 1/\sqrt{9} = 0.33$

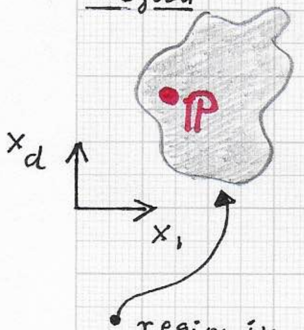
$0.5N = 6$ | $\Delta_0 = 1/\sqrt{6} = 0.41$

• Ex.: [left] given an N-point feature point set, determine minimal distance values between new feature point and the shown three sets; extrapolate for $\Delta_\infty = 0$.

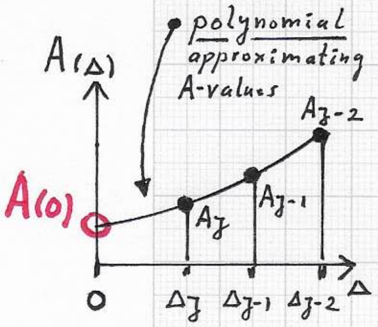
■ RICHARDSON EXTRAPOLATION FOR FEATURE SPACE – Cont'd.

...(viii) "Big picture": One has N points in a d-dimensional feature space, defined by N material samples belonging to the same class. One has one additional feature point and must decide whether it also 'represents' the same (material) class or not. Several sub-sets of the N-point data set are randomly generated – sub-sets with increasingly fewer points, e.g., $N_1, N_2, N_3, \ldots$ points. Concerning the (average) point spacing of these feature point sets, one can associate spacing values $\Delta_J, \Delta_{J-1}, \Delta_{J-2}, \Delta_{J-3}, \ldots$ with the N-point data set and sub-sets with $N_1, N_2, N_3, \ldots$ points, respectively. Thus, the N-point data set has the smallest spacing value, and spacing values increase (not satisfying a specific progression of values) monotonically up to the sub-sets with the smallest number of points. One can compute the minimal DISTANCE VALUES between the single feature point to be classified and the points in the point sets with spacings $\Delta_J, \Delta_{J-1}, \Delta_{J-2}, \Delta_{J-3}, \ldots$ These distance values can serve as distance APPROXIMATIONS $A_J, A_{J-1}, A_{J-2}, A_{J-3}, \ldots$, ultimately used to compute the 'limit distance' $A_\infty = A(\Delta=0)$, i.e., the value of a POLYNOMIAL approximation evaluated for the 'limit spacing' $\Delta=0$.

"class region"



region in d-dim. feature space with limit spacing: $\Delta=0$, i.e., infinitely dense packing



IF $|A(0)| \leq \varepsilon$ THEN POINT P BELONGS TO CLASS.

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