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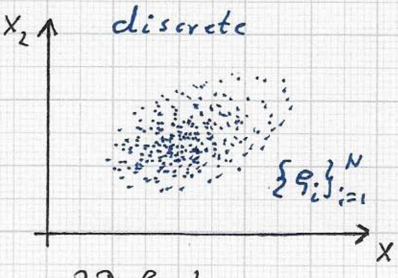
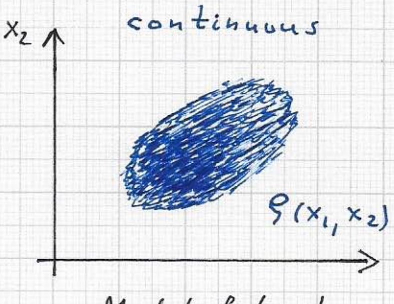
■ GEOMETRIC MODELING AND DENSITY APPROXIMATION

• Motivation: Consider a simple 2D feature space used for data classification. Left example: Whenever an element/part of an object (e.g., a voxel in a 3D volume data set) has a feature tuple (x_1, x_2) associated with it, a blue dot '$\cdot$' is made in the 2D feature point/vector space. The blue dot point cloud can be used to define a continuous approximation, a "probability density distribution function" ("density") – a MODEL for the blue dot point cloud. The model can serve two main purposes: (i) probabilistic classification of a 'new' tuple (x_1, x_2); (ii) synthetic discrete data set generation with this density.



2D feature space



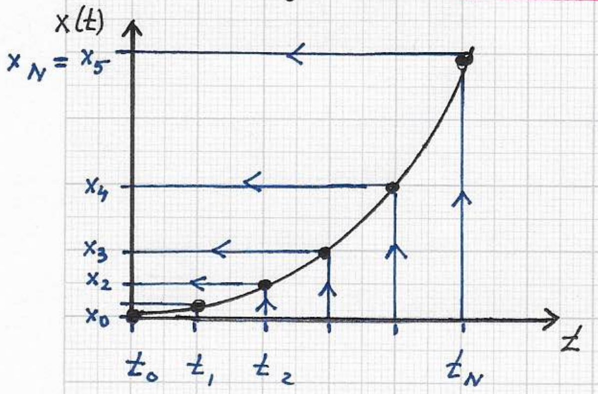
Model of density

➔ KERNEL-based smoothing/convolution $\Rightarrow \rho(x_1, x_2)$

• Issues: It is desirable to have a 'compact' analytical definition and representation of a density model. Considering the case of a high-dimensional feature space, an analytical model representation should be as efficient as possible: computational efficiency, data storage complexity and algorithmic 'simplicity' must be considered.

■ GEOMETRIC MODELING AND DENSITY – Cont'd.

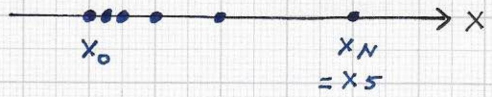
• Ex.: In numerical grid generation, for example, a continuous 'distribution function' is used to generate discrete data (for computer simulations).



distribution function

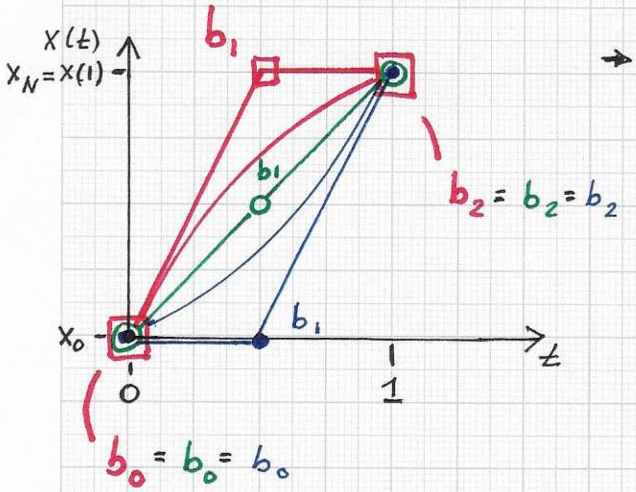
(sampled for equidistant t_i's)

⇒ resulting point distribution in 'physical' x-space:



➔ **Feature space modeling of a density function has discrete data as input; output is a continuous model.**

• Ex.: Consider the simple case of using a single quadratic distribution function (in Bernstein-Bézier form):



quadratic distributions

→ here: 3 'control points' (=coefficients) define the quadratic point distribution function:

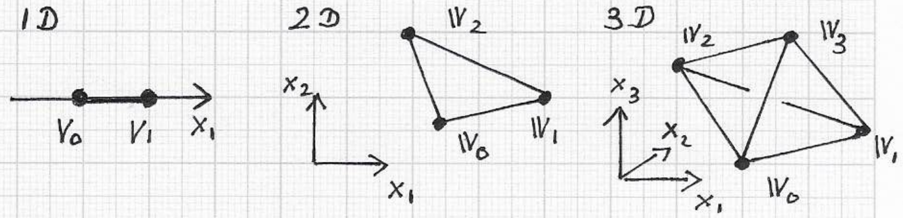
$$x(t) = \sum_{j=0}^{2} b_j B_j^2(t),$$

where $B_j^2(t) = \binom{2}{j}(1-t)^{2-j} t^j$.

⇒ In order to model complicated point distribution data one usually needs to us multiple 'polynomial segments.'

■ GEOMETRIC MODELING AND DENSITY – Cont'd.

• **SIMPLEX:** Using a simplex as 'building block' to (locally) approximate a density function over an n-dimensional feature space domain provides a high degree of efficiency, simplicity and adaptive placement. For example, consider the case of a quadratic function defined over a simplicial domain:



Geometry entirely defined by vertices:
D = dimension
n_v = no. vertices
n_e = no. edges
$\Rightarrow (D, n_v, n_e)$:
$(1,2,1)$,
$(2,3,3)$,
$(3,4,6)$,
$(4,5,10)$, ...
$\Rightarrow n_e = \frac{1}{2} D(D+1)$

Corresponding quadratic functions:

$$q(x) = c_0 + c_1 x_1 + c_2 x_1^2$$

$$q(x_1,x_2) = c_{00} + c_{10} x_1 + c_{01} x_2 + c_{11} x_1 x_2 + c_{20} x_1^2 + c_{02} x_2^2$$

$$q(x_1,x_2,x_3) = c_{000} + c_{100} x_1 + c_{010} x_2 + c_{001} x_3 + c_{110} x_1 x_2 + c_{101} x_1 x_3 + c_{011} x_2 x_3 + c_{200} x_1^2 + c_{020} x_2^2 + c_{002} x_3^2$$

$$\underline{\underline{q(x_1,\dots,x_D)}} = \underbrace{c_{i_1,\dots,i_D}}_{=0,\dots,0} + \dots + \underbrace{c_{i_1,\dots,i_D}}_{=0,\dots,0,2} x_D^2$$

$$= \sum_{\substack{i_1+\dots+i_D \le 2 \\ i_1,\dots,i_D \ge 0}} c_{i_1,\dots,i_D}\, x_1^{i_1} \cdot \ldots \cdot x_D^{i_D}$$

$$= \underline{\underline{\sum_{\|\mathbf{i}\| \le 2} c_{\mathbf{i}}\, \mathbf{x}^{\mathbf{i}}}}$$

→ No. of edges and no. of coefficients grow 'only' quadratically w.r.t. dimension.

Algebra defined by coefficients:

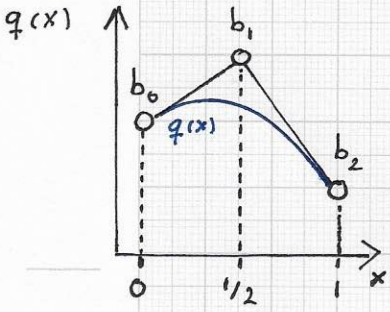
n_c = no. coefficients $\Rightarrow (D, n_c)$: $(1,3)$, $(2,6)$, $(3,10)$, $(4,15)$, ...

$\Rightarrow n_c = \frac{1}{2}(D+1)(D+2)$

GEOMETRIC MODELING AND DENSITY - Cont'd.

• SIMPLEX: Instead of using the basis functions $1, x_1, \ldots, x_D, x_1x_2, \ldots, x_{D-1}x_D, x_1^2, \ldots, x_D^2$, one can also use the quadratic Bernstein polynomials for simplicial domains. One can combine the representation of an approximating density function with an additional 'deformation' of the domain space over which the density needs to be approximated. Consider two examples: (i) a quadratic function in the Bernstein basis (to represent a density), and (ii) a parametric Bernstein-Bézier curve, combining domain space deformation and density representation over the deformed domain space.

→ $q(x) = \sum_{j=0}^{2} b_j B_j^2(x)$



$B_j^2(x) = \binom{2}{j}(1-x)^{2-j} \cdot x^j$, $x \in [0,1]$

→ Vertices $(\frac{j}{2}, b_j)^T$, $j=0,1,2$, define a 'control polygon' of $q(x)$'s graph.

→ $\{b_i\}$ = set of Bézier ordinates

(i) In this case, performing a basis transformation from the monomial to the Bernstein basis, the coefficients c_i are mapped to the coefficients b_i – but the approximation itself remains the same. For example, in the 1D case one uses the monomial basis $\{1, x, x^2\}$ and Bernstein basis $\{(1-x)^2, 2(1-x)x, x^2\}$, $x \in [0,1]$. The resulting coefficients b_0, b_1, b_2 are called 'Bézier ordinates' and define the 'control polygon' with vertices $(0, b_0)^T$, $(1/2, b_1)$, $(1, b_2)$. Note: The 'Bézier ordinates' are NOT 'control points'; they are merely scalar values, i.e., coefficients in $q(x) = b_0 B_0^2(x) + b_1 B_1^2(x) + b_2 B_2^2(x)$.

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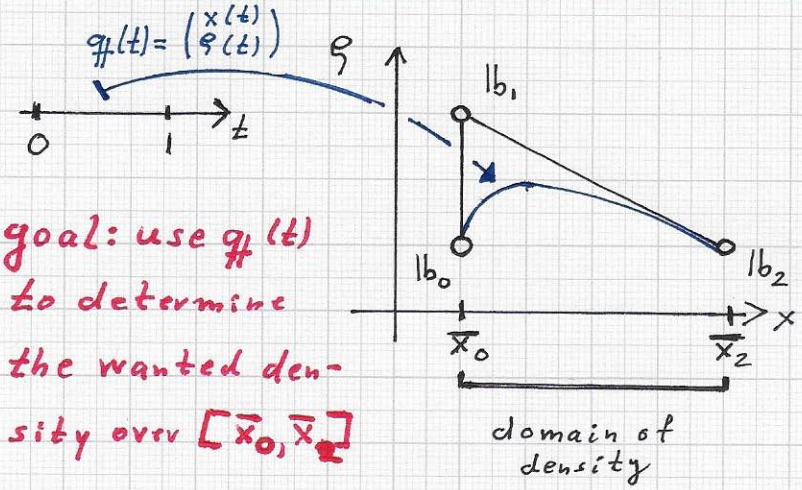
■ GEOMETRIC MODELING AND DENSITY - Cont'd.

... (ii) In this case, one defines a (quadratic) density function via a 'true' parametric Bernstein-Bézier curve:



→ $\mathbb{q}(t) = \sum_{j=0}^{2} \mathbb{b}_j B_j^2(t)$, $t \in [0,1]$

→ Vertices $(\bar{x}_j, \bar{\varrho}_j)^T$, $j = 0,1,2$, define the control polygon of the 2D-embedded curve $(x(t), \varrho(t))^T$.

• goal: use $\mathbb{q}(t)$ to determine the wanted density over $[\bar{x}_0, \bar{x}_2]$

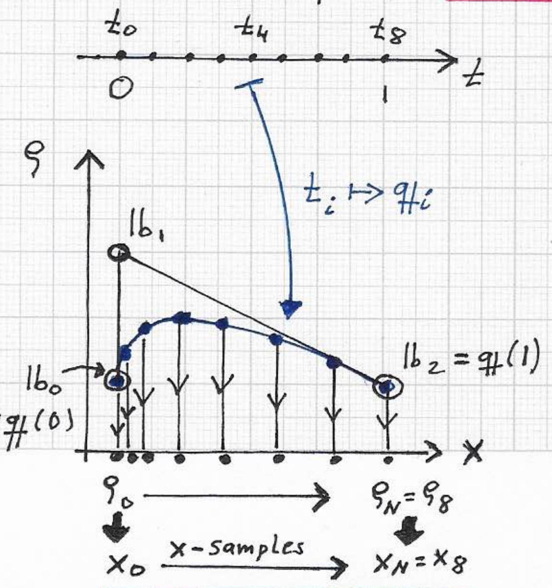
• conditions: $\bar{x}_0 < \bar{x}_2$, $\bar{x}_1 \geq \bar{x}_0$, $\bar{x}_1 \leq \bar{x}_2$, $\bar{\varrho}_j \geq 0$

→ Use **equidistantly** spaced t-values to evaluate $\mathbb{q}(t)$:

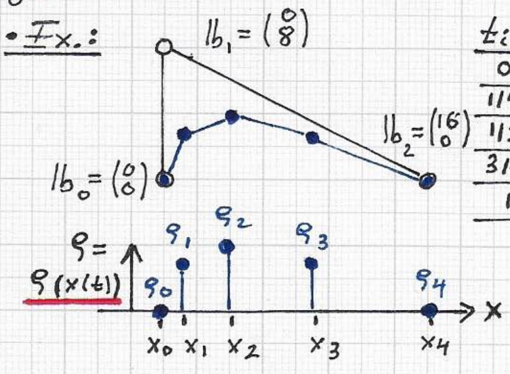
$t_i \mapsto \mathbb{q}(t_i) = \begin{pmatrix} x(t_i) \\ \varrho(t_i) \end{pmatrix}$, $i = 0 \dots N$ "points on curve"

→ $\begin{pmatrix} x(t_i) \\ \varrho(t_i) \end{pmatrix} = \begin{pmatrix} x_i \\ \varrho_i \end{pmatrix} = \mathbb{q}_i$

→ 'Project' the points $(x_i, \varrho_i)^T$ onto x-axis, defining a sample point density over the domain space, $[\bar{x}_0, \bar{x}_2]$:



• Ex.:



t_i	x_i	ϱ_i
0	0	0
1/4	1	3
1/2	4	4
3/4	9	3
1	16	0

⇒ **Coordinate function $x(t)$ defines an 'x-axis distribution' of x-samples x_i; function $\varrho(t)$ defines the density $\varrho(x(t))$.**