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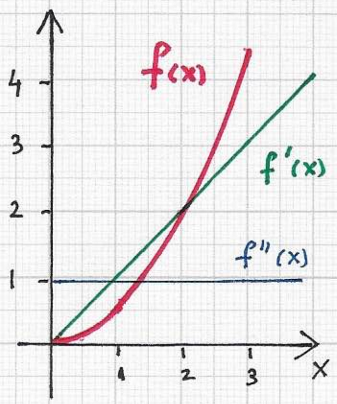
FRACTIONAL CALCULUS, FRACTIONAL DERIVATIVES ETC. AND GENERALIZED DERIVATIVE-BASED FEATURES

• Fractional calculus:



$$f(x) = \frac{d^0}{dx^0} f(x) = \frac{x^2}{2}$$

$$f'(x) = \frac{d^1}{dx^1} f(x) = x$$

$$f''(x) = \frac{d^2}{dx^2} f(x) = 1$$

Integer-order derivatives.

Generalization:

$$\boxed{\frac{d^\alpha}{dx^\alpha} f(x) = ?}$$

$\alpha \in \mathbb{R}$

Fractional derivatives allow one to smoothly "morph" the graph of $f(x)$ to the one of $f''(x)$.

Many "traditional" operators used in calculus are based on integers: Examples are $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$, $n \in \mathbb{N}$, or the derivatives of a function $f(x)$, i.e., $f'(x), f''(x), \ldots, f^{(n)}(x) = \frac{d^n}{dx^n} f(x)$, $n \in \mathbb{N}$. Considering the derivatives of a function, it is possible to generalize the traditional integer-order derivative operator to a so-called "fractional-order derivative" operator, e.g., one can define $f^{(1/2)}(x)$. The generalization to fractional derivatives can be done in several possible ways – as long as the generalization agrees with the traditional derivative $\frac{d^n}{dx^n} f(x)$ for integer values of n. Why is this of interest and relevance? In our context, i.e., 2D and 3D image/scan processing and classification, we rely on powerful means to define and compute features. Features are commonly related to and determined based on image/scan derivatives. Thus, fractional derivatives are very promising for classification!

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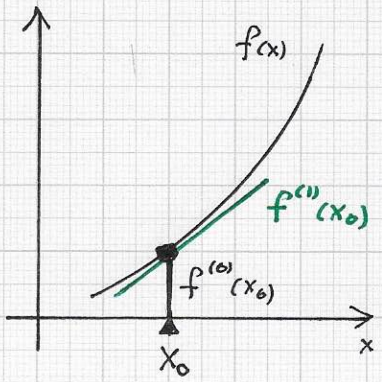
■ FRACTIONAL CALCULUS AND FEATURES – Cont'd.

• **Fractional calculus:**



Motivation for a generalization of traditional integer-order calculus operators:

POWER (TAYLOR) SERIES OF A FUNCTION:

INTEGER CALCULUS

$$\underline{f(x)} = \frac{f^{(0)}(x_0)}{0!}\cdot(x-x_0)^0 + \frac{f^{(1)}(x_0)}{1!}\cdot(x-x_0)^1 + \frac{f^{(2)}(x_0)}{2!}\cdot(x-x_0)^2 + \ldots + \frac{f^{(n)}(x_0)}{n!}\cdot(x-x_0)^n = \sum_{i=0}^{n}\frac{f^{(i)}(x_0)}{i!}\cdot(x-x_0)^i$$

To a large degree, traditional calculus was developed to mathematically describe physical phenomena – position, velocity, acceleration etc. Thus, integer-order calculus is "mapped" to physically observed behavior. Since it is possible to generalize integer-order calculus operators to fractional-order operators, the question is whether a more complicated fractional-order operator is "meaningful" or "helpful" for applications. By now it has become evident that fractional-order calculus operators make it possible to generalize traditional 2D/3D/4D image processing and analysis operators, i.e., they allow one to extend the well-known first and second derivative operators for edge/boundary extraction or texture characterization. In the context of material/object classification, the use of features based on fractional derivative behavior, for example, provides many non-traditional possibilities for defining feature points/vectors. This fact should greatly improve material/object classification via fractional derivative features.

FRACTIONAL CALCULUS AND FEATURES – Cont'd.

• Fractional calculus:

→ Traditional calculus:

Using integer-order derivatives at a location x_0 to characterize the function's behavior LOCALLY; usually considering only

$f^{(0)}(x_0), \dots, f^{(4)}(x_0)$

→ Fractional calculus:

Using fractional-order derivatives

$d^\alpha / dx^\alpha \, f(x_0), \alpha \in \mathbb{R}$,

that characterize the function's behavior not only locally but for an entire RANGE or NEIGHBORHOOD in the x-domain.

⇒ ONE CAN COMPUTE AND USE A MULTITUDE OF FRACTIONAL DERIVATIVE FEATURE DATA TO OPTIMIZE IMAGE DATA CLASSIFICATION.

For example, the traditional integer-order power (Taylor) series of a function $f(x)$ with integer-order derivatives $f^{(i)}(x_0)$ is $f(x) = \sum_{i=0}^{n} \left(f^{(i)}(x_0)/i!\right) \cdot (x-x_0)^i$; it involves the factorial operator '!' applied to integer i and the i^{th} derivative of f at location x_0. As just one possibility, one can generalize integer-order calculus to fractional-order calculus by extending such a power series, motivating the need for (i) extending the factorial operator '!' to non-integer numbers and (ii) extending the derivative operator $\frac{d^i}{dx^i} f(x)$, $i \in \mathbb{N}$, to non-integer numbers, leading to $\frac{d^\alpha}{dx^\alpha} f(x)$, $\alpha \in \mathbb{R}$.

The common, standard generalization of the factorial operator '!' is the Gamma function $\Gamma(z)$ defined for a complex number z. For our purposes it suffices to only consider real arguments of the Gamma function:

$$\boxed{\Gamma(x) = \int_{t=0}^{\infty} e^{-t}\, t^{x-1}\, dt} \quad , x \geq 1.$$

• Note: → $\Gamma(i) = (i-1)!$, $i \in \mathbb{N}$.

→ $\Gamma(x+1) = x\,\Gamma(x)$, $x \geq 1$.

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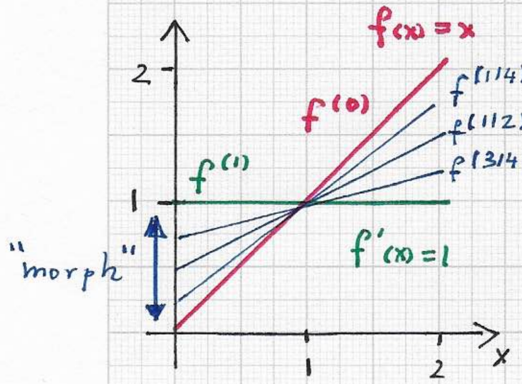
FRACTIONAL CALCULUS AND FEATURES – Cont'd.

• Fractional calculus:



Smooth, continuous transformation of graphs of original function $f(x)$ and its first derivative $f'(x)$.

The goal is to define fractional derivatives $f^{(\alpha)}(x)$, $0<\alpha<1$, in such a way that the graphs, the geometry, of $f(x)$ and $f'(x)$ are smoothly morphed via the fractional derivatives "between" $f(x)$ and $f'(x)$.

(The illustration does NOT depict the specific behavior of any of the common definitions of a fractional derivative.)

⇒ FRACTIONAL DERIVATIVES CAN BE DEFIND IN VARIOUS MEANINGFUL WAYS!

When formally applying differentiation rules to the power function $f(x) = x^y$, $x \geq 0$, one obtains

$$\frac{d^\alpha}{dx^\alpha}\left(x^y\right) = \frac{y!}{(y-\alpha)!}\,x^{y-\alpha} = \frac{\Gamma(1+y)}{\Gamma(1+y-\alpha)}\,x^{y-\alpha}.$$

Most importantly, when dealing with polynomials $pol(x) = p(x) = \sum_{i=0}^{n} c_i x^i$, or when one knows the power series of a function, one can employ this rule to compute all α-order derivatives of all terms. In this case, the more specific rule is the rule for $f(x) = x^n$, $n \in \mathbb{N}_0$, i.e.,

$$\frac{d^\alpha}{dx^\alpha}\left(x^n\right) = \frac{\Gamma(1+n)}{\Gamma(1+n-\alpha)}\,x^{n-\alpha}.$$

• **Note:** Since most 2D/3D/4D image analysis methods use finite difference formulas for local polynomial approximations of given image intensity data, the fractional derivative $d^\alpha/dx^\alpha\,(pol(x))$ is most relevant for derivative-based features of image and scan data.

• Ex.: $e^x \overset{x_0=0}{=} \sum_{i=0}^{\infty} \frac{e^0}{i!}\,x^i = \frac{1}{0!}x^0 + \frac{1}{1!}x^1 + \frac{1}{2!}x^2 + \ldots$

$$\Rightarrow \frac{d^\alpha}{dx^\alpha}\,e^x = \sum_{i=0}^{\infty} \frac{1}{i!}\,\frac{d^\alpha}{dx^\alpha}\,x^i = \sum_{i=0}^{\infty} \frac{1}{\Gamma(1+i)}\,\frac{\Gamma(1+i)}{\Gamma(1+i-\alpha)}\,x^{i-\alpha}$$

$$= \sum_{i=0}^{\infty} \frac{1}{\Gamma(1+i-\alpha)}\,x^{i-\alpha}. \quad \text{(Riemann)}$$

FRACTIONAL CALCULUS AND FEATURES – Cont'd.

• Fractional calculus: Fractional differentiation and integration can also be defined in SEVERAL WAYS for a general function $f(x)$. (While the traditional integer-order operators compute LOCAL properties of $f(x)$, many fractional-order generalizations compute non-local properties.) The most common fractional-order generalizations are:

- the RIEMANN-LIOUVILLE fractional INTEGRAL;
- the RIEMANN-LIOUVILLE fractional DERIVATIVE;
- the GRÜNWALD-LETNIKOV fractional DERIVATIVE; and
- the CAPUTO fractional DERIVATIVE.

RL fractional integration & differentiation defined via interval-based operators $\mathcal{I}$ & $\mathcal{D}$, considering interval from a to x ($a<x$) or x to b ($x<b$) and using fractional order α

(i) Riemann-Liouville (RL) integrals for an INTERVAL $[a,b]$:

$$ {}_a\mathcal{D}_x^{-\alpha} f(x) = {}_a\mathcal{I}_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^{x} (x-\tilde{x})^{\alpha-1} f(\tilde{x})\, d\tilde{x} \quad (x>a) $$

$$ {}_x\mathcal{D}_b^{-\alpha} f(x) = {}_x\mathcal{I}_b^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^{b} (\tilde{x}-x)^{\alpha-1} f(\tilde{x})\, d\tilde{x} \quad (x<b) $$

(ii) RL derivatives, $n=\lceil\alpha\rceil$ (smallest int. $>\alpha$):

$$ {}_a\mathcal{D}_x^{\alpha} f(x) = \frac{d^n}{dx^n}\, {}_a\mathcal{I}_x^{n-\alpha} f(x) $$

$$ {}_x\mathcal{D}_b^{\alpha} f(x) = \frac{d^n}{dx^n}\, {}_x\mathcal{I}_b^{n-\alpha} f(x) $$

• Note: The Gamma function $\Gamma(\alpha)$ is not defined for negative real numbers; $(n-\alpha)$ is positive.