FRACTIONAL CALCULUS AND FEATURES - Cont'd.

• Fractional calculus:

(iii) Caputo derivative, $n = \lceil \alpha \rceil$:

$$D_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-\tilde{x})^{n-\alpha-1} f^{(n)}(\tilde{x})\, d\tilde{x}$$

The CAPUTO derivative is local and is 0 for a constant function.

• Note: Unlike the RL derivative, the Caputo derivative is zero for a constant function, and the Caputo derivative is Local.

(iv) RL derivative for intervals $[a,x]$ and $[x,b]$ - slightly rewritten:

$$_aD_x^\alpha f(x) = \frac{d^n}{dx^n}\left(\frac{1}{\Gamma(n-\alpha)} \int_a^x (x-\tilde{x})^{n-\alpha-1} f(\tilde{x})\, d\tilde{x}\right)$$

$$_xD_b^\alpha f(x) = \frac{d^n}{dx^n}\left(\frac{1}{\Gamma(n-\alpha)} \int_x^b (x-\tilde{x})^{n-\alpha-1} f(\tilde{x})\, d\tilde{x}\right)$$

RL derivatives are based on "time" intervals $[a,x]$ and $[x,b]$ and are not 0 for a constant function.

• Note: In the context of a time-dependent "signal" $f(x)$, considering the interval $[a,x]$ in the computation of the fractional derivative is motivated by the thought that derivative behavior observed depends on a time interval $[a,x]$ and not just time point x.

• SINCE ONE CAN DEFINE FRACTIONAL DERIVATIVES IN MANY WAYS, A "GOOD DEFINITION" IS ONE THAT SOLVES A PROBLEM WELL.

FRACTIONAL CALCULUS AND FEATURES – Cont'd.

• Fractional derivative:

Given: 2D/3D image/scan of intensity values

Compute:
- Local polynomial approximations
- 1^{st} and 2^{nd} derivative data...
- ADDITIONAL FRACTIONAL DERIVATIVES OF ORDER α, $0 < \alpha < 2$

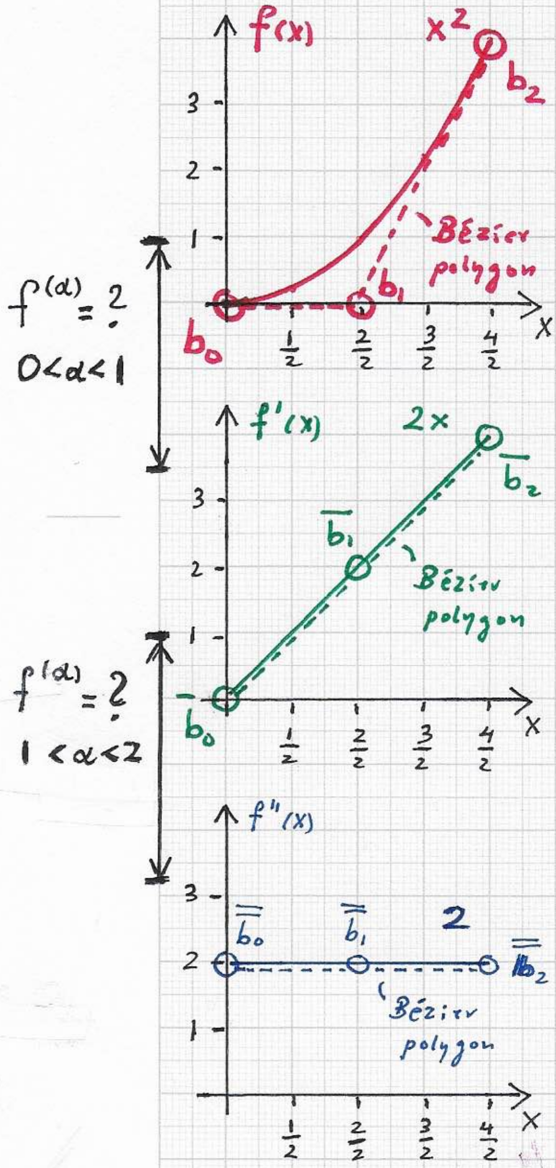
FILTER MASKS: DEVISE AND USE DISCRETE FILTER MASKS TO COMPUTE FRACTIONAL DERIVATIVES DIRECTLY FROM PIXEL/VOXEL DATA!

Our goal is the definition of features of 2D/3D/4D images and scans that support an increasingly accurate classification of objects/materials. Typically, the given data are intensity data representing object/material densities. Standard image processing and analysis methods consider intensity values, first derivative estimates, second derivative estimates etc. to characterize a material (its density, its boundary, its "texture"...). Fractional calculus makes it possible to define and compute "many fractional derivative-based feature data," e.g., "derivatives between the 0^{th} and 1^{st} derivative" or "derivatives between the 1^{st} and 2^{nd} derivative." Such additional data, used to define more feature points/vectors, should make possible improved classification of objects/materials that are very similar.

For example, the RL and Caputo fractional derivatives can be used to compute fractional derivatives of local polynomial image/scan intensity approximations. Further, one can determine the local pixel/voxel neighborhood masks to compute these derivatives.

■ FRACTIONAL CALCULUS AND FEATURES – Cont'd.

• Fractional derivative:



Integer-order derivatives of $f(x)=x^2$. All 3 polynomial derivatives raised to degree 2, i.e., each derivative represented with 3 polynomial coefficients, $\{b_i\}$, $\{\bar{b}_i\}$, $\{\bar{\bar{b}}_i\}$, $i=0\ldots2$.

Fractional derivative operators (RL, Caputo etc.) can be viewed as operators that smoothly blend from 0^{th} derivative to 1^{st} derivative, or from 1^{st} derivative to 2^{nd} derivative. Since the traditional fractional derivative operators are relatively "expensive" to compute, one can explore the possibility to simply perform piecewise linear blending between the integer-order derivatives of a function. This could be done, for example, by representing the integer-order derivatives of a (polynomial) function in Bernstein-Bézier form and blending the Bernstein-Bézier coefficients ("control points") in a piecewise linear fashion when blending i^{th} and $(i+1)^{th}$ derivatives. (Each polynomial integer-order derivative can be represented with the same number of control points / coefficients via degree raising.)

The viability of such a blending method can be determined by comparing this method's blended results to those of the RL or Caputo results.

■ FRACTIONAL CALCULUS AND FEATURES – Cont'd.

• **Fractional derivative:** Ex.: CAPUTO derivatives of $f(x)=1$, $f(x)=x$, $f(x)=x^2$; fractional orders $0, \frac{1}{2}, 1, \frac{3}{2}, 2$

[abbreviated notation: $f^{(\alpha)}(x) = D_x^\alpha f(x)$]

CAPUTO derivatives of low-degree polynomials

(i) $f(x)=1$

$$f^{(\alpha)}(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-\tilde{x})^{n-\alpha-1} \cdot f^{(n)}(\tilde{x})\, d\tilde{x}$$

$$= \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-\tilde{x})^{n-\alpha-1} \cdot 0 \, d\tilde{x} = 0$$

(ii) $f(x)=x$

⊛ $\alpha=0$: $$f^{(0)} = \frac{1}{\Gamma(1)} \int_0^x (x-\tilde{x})^0 \cdot f^{(1)}(\tilde{x})\, d\tilde{x}$$

$$= \frac{1}{\Gamma(1)} \int_0^x 1 \cdot 1 \, d\tilde{x} = x$$

$\alpha=\frac{1}{2}$: $$f^{(1/2)} = \frac{1}{\Gamma(1/2)} \int_0^x (x-\tilde{x})^{-1/2} \cdot f^{(1)}(\tilde{x})\, d\tilde{x}$$

$$= \frac{1}{\Gamma(1/2)} \int_0^x (x-\tilde{x})^{-1/2} \cdot 1 \, d\tilde{x}$$

$$= \frac{1}{\Gamma(1/2)} (-2) \cdot (x-\tilde{x})^{1/2} \Big|_0^x$$

$$= \frac{2}{\Gamma(1/2)} x^{1/2} = \frac{2}{\sqrt{\pi}} x^{1/2} = 2\sqrt{x/\pi}$$

⊛ $\alpha=1$: $$f^{(1)} = \frac{1}{\Gamma(1)} \int_0^x (x-\tilde{x})^0 \underbrace{f^{(2)}(\tilde{x})}_{=0} d\tilde{x}$$

$$= \frac{1}{\Gamma(1)} \int_0^x 1 \cdot 0 \, d\tilde{x} = 0 \quad \textbf{NO! } f^{(1)} = 1$$

$\alpha=\frac{3}{2}$: $$f^{(3/2)} = \frac{1}{\Gamma(1/2)} \int_0^x (x-\tilde{x})^{-1/2} \cdot \underbrace{f^{(2)}(\tilde{x})}_{=0} d\tilde{x} = 0$$

⊛ $\alpha=2$: $$f^{(2)} = \frac{1}{\Gamma(1)} \int_0^x (x-\tilde{x})^0 \cdot \underbrace{f^{(3)}(\tilde{x})}_{=0} d\tilde{x} = 0$$

• **Note:** Use WOLFRAM ALPHA, for example, to compute values of Gamma function and integrals.

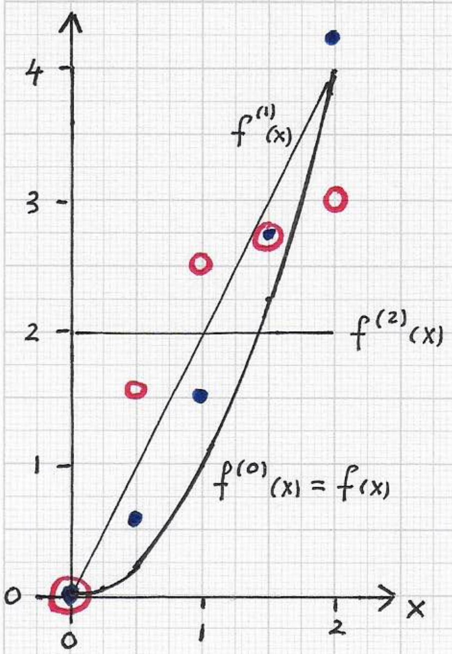
⊛ Use standard integer-order differentiation rules when $\alpha \in \mathbb{N}$ ⇒ avoiding degenerate cases!

■ FRACTIONAL CALCULUS AND FEATURES – Cont'd.

• Fractional derivative: **CAPUTO derivatives ...**

CAPUTO derivatives of polynomials

$f(x) = x^2$



x	$f^{(1/2)}$	$f^{(3/2)}$
0	0	0
1/2	0.53	1.60
1	1.50	2.56
3/2	2.76	2.76
2	4.26	3.19

$\uparrow \{\bullet\}$ $\qquad$ $\uparrow \{\circ\}$

(iii) $f(x) = x^2$

$\alpha = 0$: $$f^{(0)} = \frac{1}{\Gamma(1)} \int_0^x (x-\tilde{x})^0 f^{(1)}(\tilde{x})\, d\tilde{x} = \frac{1}{\Gamma(1)} \int_0^x 2\tilde{x}\, d\tilde{x} = \mathbf{x^2}$$

$\alpha = \frac{1}{2}$: $$f^{(1/2)} = \frac{1}{\Gamma(1/2)} \int_0^x (x-\tilde{x})^{-1/2} f^{(1)}(\tilde{x})\, d\tilde{x} = \frac{1}{\Gamma(1/2)} \int_0^x (x-\tilde{x})^{-1/2}\, 2\tilde{x}\, d\tilde{x} = \frac{2}{\sqrt{\pi}} \int_0^x (x-\tilde{x})^{-1/2}\, \tilde{x}\, d\tilde{x} = \ldots = \frac{4}{3} x^{3/2} \cdot \frac{2}{\sqrt{\pi}}$$

$(Re(x) > 0 \wedge Im(x) = 0)$

$\alpha = 1$: $$f^{(1)} = \frac{1}{\Gamma(1)} \int_0^x (x-\tilde{x})^0 f^{(2)}(\tilde{x})\, d\tilde{x} = \frac{1}{\Gamma(1)} \int_0^x 1 \cdot 2\, d\tilde{x} = \mathbf{2x}$$

$\alpha = \frac{3}{2}$: $$f^{(3/2)} = \frac{1}{\Gamma(1/2)} \int_0^x (x-\tilde{x})^{-1/2} f^{(2)}(\tilde{x})\, d\tilde{x} = \frac{1}{\Gamma(1/2)} \int_0^x (x-\tilde{x})^{-1/2} \cdot 2\, d\tilde{x} = \frac{2}{\sqrt{\pi}} \int_0^x (x-\tilde{x})^{-1/2}\, d\tilde{x} = \mathbf{4\sqrt{x/\pi}}$$

⊛ $\alpha = 2$: $$f^{(2)} = \frac{1}{\Gamma(1)} \int_0^x (x-\tilde{x})^0 \underbrace{f^{(3)}(\tilde{x})}_{=0}\, d\tilde{x} = 0$$

NO! $f^{(2)} = 2$

⊛ Use integer-order differentiation!

• **Note:** $f(x) = x^2$; $\alpha = 1/k$, $k = 2,3,4,\ldots$; $n = 2$

$$\Rightarrow f^{(1/k)} = \frac{2}{\Gamma(1-1/k)} \cdot \frac{k^2}{2k^2 - 3k + 1}\, x^{3/2},$$

$Re(x) > 0 \wedge Im(x) = 0$

• **Note:** $\Gamma(1/2) = \sqrt{\pi}$, $\Gamma(1) = \Gamma(2) = 1$, ...