■ FRACTIONAL CALCULUS AND FEATURES – Cont'd.

• Fractional derivatives:

$f_{i,j}^{(2,0)}$:

	$i-1$	i	$i+1$
j	1	-2	1

$f_{i,j}^{(0,2)}$:

	i
$j+1$	1
j	-2
$j-1$	1

$f_{i,j}^{(2,1)}$:

	$i-1$	i	$i+1$
$j+1$	$\frac{1}{2}$	-1	$\frac{1}{2}$
j	0	0	0
$j-1$	$-\frac{1}{2}$	1	$-\frac{1}{2}$

$f_{i,j}^{(1,2)}$:

	$i-1$	i	$i+1$
$j+1$	$-\frac{1}{2}$	0	$\frac{1}{2}$
j	1	0	-1
$j-1$	$-\frac{1}{2}$	0	$\frac{1}{2}$

Stencils and weights for partial derivatives $f_{i,j}^{(p,q)}$

2) $f_{i,j}^{(2,0)} = f_{i+1,j} - 2f_{i,j} + f_{i-1,j}$

$f_{i,j}^{(0,2)} = f_{i,j+1} - 2f_{i,j} + f_{i,j-1}$

3) $$f_{i,j}^{(2,1)} = \left(f_{i,j+1}^{(2,0)} - f_{i,j-1}^{(2,0)}\right)/2$$
$$= \big(f_{i+1,j+1} - 2f_{i,j+1} + f_{i-1,j+1} - (f_{i+1,j-1} - 2f_{i,j-1} + f_{i-1,j-1})\big)/2$$
$$= \frac{f_{i+1,j+1} - 2f_{i,j+1} + f_{i-1,j+1} - f_{i+1,j-1} + 2f_{i,j-1} - f_{i-1,j-1}}{2}$$

$$f_{i,j}^{(1,2)} = \left(f_{i+1,j}^{(0,2)} - f_{i-1,j}^{(0,2)}\right)/2$$
$$= \big(f_{i+1,j+1} - 2f_{i+1,j} + f_{i+1,j-1} - (f_{i-1,j+1} - 2f_{i-1,j} + f_{i-1,j-1})\big)/2$$
$$= \frac{f_{i+1,j+1} - 2f_{i+1,j} + f_{i+1,j-1} - f_{i-1,j+1} + 2f_{i-1,j} - f_{i-1,j-1}}{2}$$

4) $f_{i,j}^{(2,2)}$ = "result of applying the $f_{i,j}^{(2,0)}$ operator 3 times, to compute $f_{i,j-1}^{(2,0)}$, $f_{i,j}^{(2,0)}$ and $f_{i,j+1}^{(2,0)}$, and applying the $f_{i,j}^{(0,2)}$ operator once, to compute $f_{i,j}^{(2,2)}$"

$$= -2f_{i,j+1} + f_{i+1,j+1} + f_{i-1,j+1} - 2(-2f_{i,j} + f_{i+1,j} + f_{i-1,j}) - 2f_{i,j-1} + f_{i+1,j-1} + f_{i-1,j-1}$$

$$= f_{i+1,j+1} - 2f_{i,j+1} + f_{i-1,j+1}$$
$$- 2f_{i+1,j} + 4f_{i,j} - 2f_{i-1,j}$$
$$+ f_{i+1,j-1} - 2f_{i,j-1} + f_{i-1,j-1}$$

: $f_{i,j}^{(2,2)}$

	$i-1$	i	$i+1$
$j+1$	1	-2	1
j	-2	4	-2
$j-1$	1	-2	1

FRACTIONAL CALCULUS AND FEATURES – Cont'd.

• **Fractional derivatives:**

We also require the finite difference formulas for the approximation of integer-order derivatives at the center of a voxel (i,j,k), with unit edge length for a simplified spacing $\Delta = 1$ in all three directions: $f_{i,j,k}^{(p,q,r)}$

0) $f_{i,j,k}^{(0,0,0)} = f_{i,j,k}$

1) $f_{i,j,k}^{(1,0,0)} = \dfrac{f_{i+1,j,k} - f_{i-1,j,k}}{2}$, $f_{i,j,k}^{(0,1,0)} = \dfrac{f_{i,j+1,k} - f_{i,j-1,k}}{2}$,

$f_{i,j,k}^{(0,0,1)} = \dfrac{f_{i,j,k+1} - f_{i,j,k-1}}{2}$

$f_{i,j,k}^{(1,1,0)} = \dfrac{f_{i+1,j+1,k} - f_{i-1,j+1,k} - f_{i+1,j-1,k} + f_{i-1,j-1,k}}{4}$

$f_{i,j,k}^{(1,0,1)} = \dfrac{f_{i+1,j,k+1} - f_{i-1,j,k+1} - f_{i+1,j,k-1} + f_{i-1,j,k-1}}{4}$

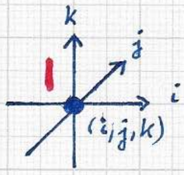
$f_{i,j,k}^{(0,1,1)} = \dfrac{f_{i,j+1,k+1} - f_{i,j-1,k+1} - f_{i,j+1,k-1} + f_{i,j-1,k-1}}{4}$

2) $f_{i,j,k}^{(2,0,0)} = f_{i+1,j,k} - 2 f_{i,j,k} + f_{i-1,j,k}$

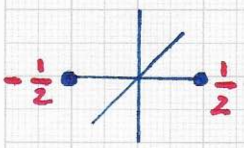
$f_{i,j,k}^{(0,2,0)} = f_{i,j+1,k} - 2 f_{i,j,k} + f_{i,j-1,k}$

$f_{i,j,k}^{(0,0,2)} = f_{i,j,k+1} - 2 f_{i,j,k} + f_{i,j,k-1}$

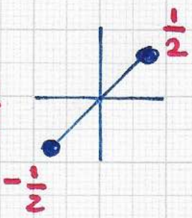
$f_{i,j,k}^{(0,0,0)}$:



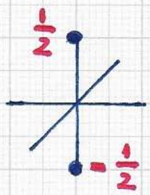
$f_{i,j,k}^{(1,0,0)}$:



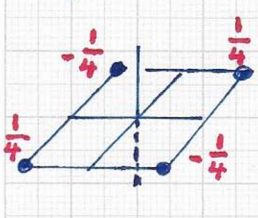
$f_{i,j,k}^{(0,1,0)}$:



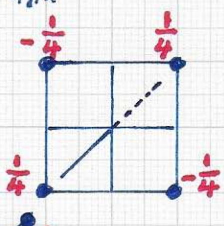
$f_{i,j,k}^{(0,0,1)}$:



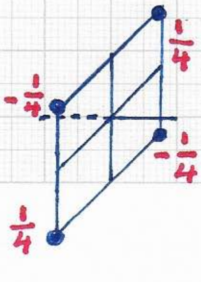
$f_{i,j,k}^{(1,1,0)}$:



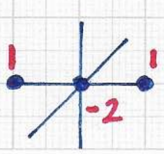
$f_{i,j,k}^{(1,0,1)}$:



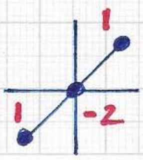
$f_{i,j,k}^{(0,1,1)}$:



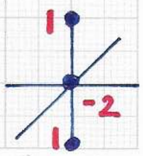
$f_{i,j,k}^{(2,0,0)}$:



$f_{i,j,k}^{(0,2,0)}$:



$f_{i,j,k}^{(0,0,2)}$:



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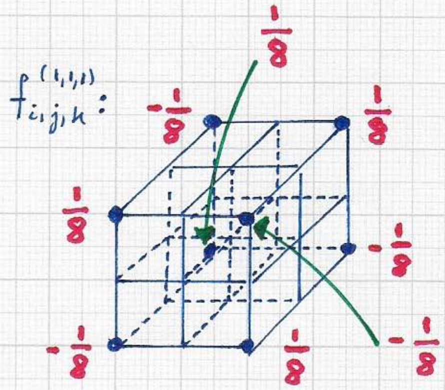
8/14/2021

■ FRACTIONAL CALCULUS AND FEATURES - Cont'd.

• Fractional derivatives:

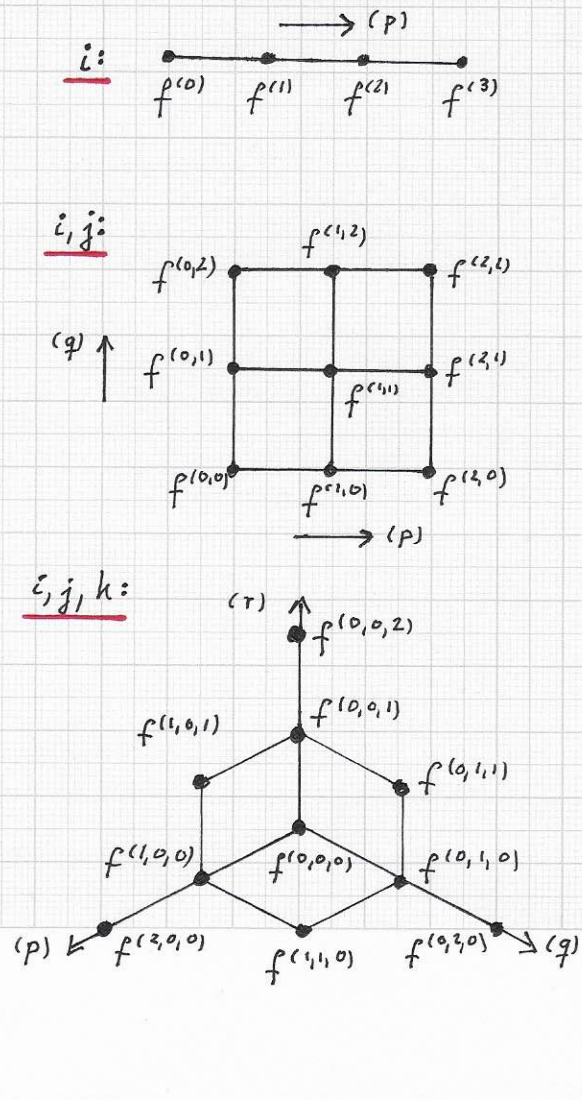
3)
$$f^{(1,1,1)}_{i,j,k} = \left(\frac{f_{i+1,j+1,k+1} - f_{i-1,j+1,k+1} - f_{i+1,j-1,k+1} + f_{i-1,j-1,k+1}}{4} - \frac{f_{i+1,j+1,k-1} - f_{i-1,j+1,k-1} - f_{i+1,j-1,k-1} + f_{i-1,j-1,k-1}}{4} \right) \Big/ 2$$

$$= \frac{1}{8} \left(f_{i+1,j+1,k+1} - f_{i-1,j+1,k+1} - f_{i+1,j-1,k+1} + f_{i-1,j-1,k+1} - f_{i+1,j+1,k-1} + f_{i-1,j+1,k-1} + f_{i+1,j-1,k-1} - f_{i-1,j-1,k-1} \right)$$



4) $f^{(2,1,0)}_{i,j,k} = \ldots$, $f^{(2,2,0)}_{i,j,k} = \ldots$, $f^{(2,2,2)}_{i,j,k} = \ldots$

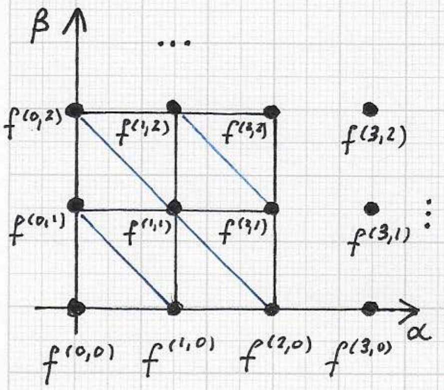
1D case | 2D case | 3D case



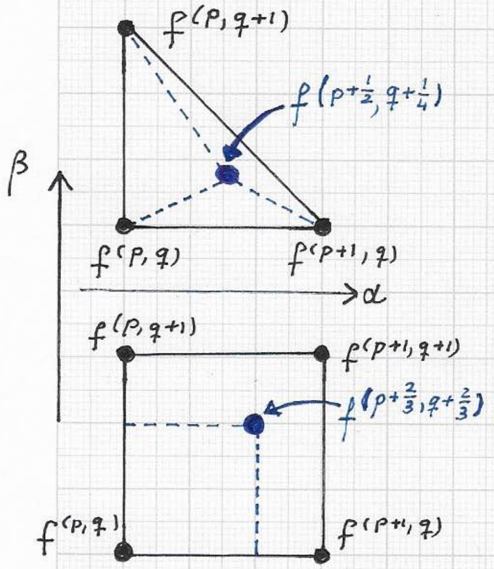
These finite difference formulas define integer-order (partial) derivative approximations for 1D, 2D and 3D "elements" with index (tuples) (i), (i,j) and (i,j,k), respectively. The (partial) derivative approximations of orders (p), (p,q) and (p,q,r), $p,q,r \in \{0,1,2,\ldots\}$, can be associated with the corresponding element. The illustrations (left) depict the low-order partial derivative stencils one can compute and store for a 1D, 2D or 3D element. These discrete stencils provide the input for the computation of non-integer-order derivatives.

FRACTIONAL CALCULUS AND FEATURES – Cont'd.

• **Fractional derivatives:**



Stencil of partial integer-order derivatives associated with one pixel. "Derivative vertices" $f^{(p,q)}$ – represented as points $(p,q)^T$ in the (α,β) domain can be connected via triangles or quadrilaterals.



Triangle/quadrilateral containing arbitrary point $(\alpha,\beta)^T$ in interior; e.g., use linear or bilinear interpolation to get $f^{(\alpha,\beta)}$.

Once small-order, integer-order (partial) derivative approximations have been computed for pixels (i,j), or voxels (i,j,k), one can use the approximations to define and compute (α,β)-order, or (α,β,γ)-order, non-integer-order derivatives (i.e., $\alpha,\beta,\gamma\in\mathbb{R}$, $\alpha,\beta,\gamma\geq 0$). One can consider to basic numerical approaches for computing these derivatives, $f_{i,j}^{(\alpha,\beta)}$ or $f_{i,j,k}^{(\alpha,\beta,\gamma)}$: (i) one can construct a simple grid connecting the given integer-order partial derivatives ("derivative vertices") (p,q) or (p,q,r) – in other words, one uses the integer-order tuples (p,q) or (p,q,r) as points in a continuous (α,β) or (α,β,γ) domain and connects the points (defining triangles, quadrilaterals or tetrahedra, hexahedra); one can then use a grid-based interpolation method to compute a partial derivative estimate for any real-number tuple (α,β) or (α,β,γ); (ii) one uses an interpolation method that does not require a grid; choosing a "proper" grid-less interpolation method is crucial.

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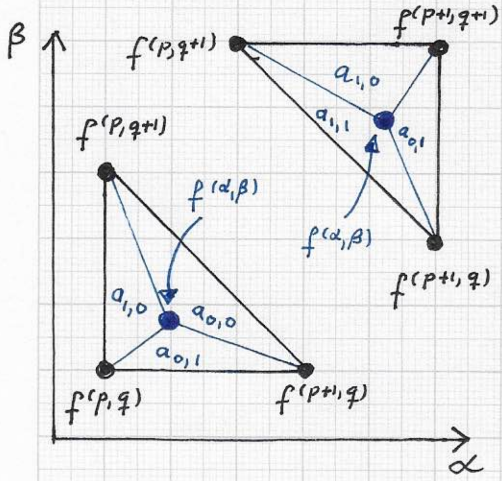
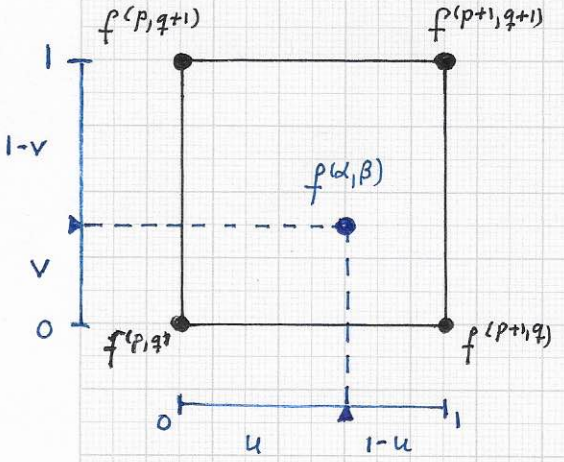
FRACTIONAL CALCULUS AND FEATURES – Cont'd.

• Fractional derivatives:



Prototypical configurations for performing linear interpolation over triangles to compute partial derivative $f^{(\alpha,\beta)}$



Prototypical configuration for performing bilinear interpolation over a unit square to compute partial derivative $f^{(\alpha,\beta)}$; local parameter values u and v given as $u = \alpha - p$ and $v = \beta - q$

Using "barycentric coordinates" for linear interpolation over a triangle is natural. The figure (left) depicts the two prototypical cases one must consider when computing a non-integer-order partial derivative $f^{(\alpha,\beta)}$ via linear interpolation of three integer-order partial derivatives given at the triangles' corners.

The sub-areas of the two prototypical triangles are called $a_{0,0}$, $a_{1,0}$ and $a_{0,1}$ (left triangle) and $a_{1,1}$, $a_{1,0}$ and $a_{0,1}$ (right triangle). Both triangles have an area of $1/2$. Thus, the value of $f^{(\alpha,\beta)}$ is given as

$$f^{(\alpha,\beta)} = 2\left(a_{0,0}\, f^{(p,q)} + a_{1,0}\, f^{(p+1,q)} + a_{0,1}\, f^{(p,q+1)}\right)$$

for the left triangle and

$$f^{(\alpha,\beta)} = 2\left(a_{1,1}\, f^{(p+1,q+1)} + a_{1,0}\, f^{(p+1,q)} + a_{0,1}\, f^{(p,q+1)}\right)$$

for the right triangle.

Bilinear interpolation of four integer-order partial derivatives given at a unit square's corners is given as

$$f^{(\alpha,\beta)} = (1-u)\cdot(1-v)\cdot f^{(p,q)} + (1-u)\cdot v\cdot f^{(p,q+1)} + u\cdot(1-v)\cdot f^{(p+1,q)} + u\cdot v\cdot f^{(p+1,q+1)},$$

where $u = \alpha - p$ and $v = \beta - q$.

~BH