■ OBJECT AND MATERIAL EIGENFUNCTION FEATURES

• Laplacian-Motivation:

Eigenfunctions, eigenvalue spectra, "eigenmodes" of a
- string,
- membrane,
- drum, ...

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Second-order partial differential equations, e.g., Helmholtz equation:

$$-\Delta f = \lambda f ,$$

$$\Delta f = \operatorname{div} \operatorname{grad} f = \nabla \cdot \nabla f = \frac{\partial^2}{\partial x^2} f + \frac{\partial^2}{\partial y^2} f + \frac{\partial^2}{\partial z^2} f = \text{"Laplacian of } f\text{"}$$

The Laplace and the more general Laplace-Beltrami operators have been used extensively to describe and computationally simulate physical phenomena and, more recently, to analyze, characterize and classify "shapes" (usually understood as complicated 1D, 2D or 3D manifold geometry data). In the context of discrete geometric data processing, it turns out that a shape represented by a manifold grid (triangles/quadrilaterals or tetrahedra/hexahedra, for example) can be characterized in terms of its "eigenfunctions" that are implied by the vertices and grid used to represent the shape discretely. More generally, it is also possible to associate a material property with the shape, e.g., one could associate a material density with each vertex (or grid cell) used to represent a 3D object with varying density.

In the context of material/object classification, the following question arises:

* DO LAPLACIAN EIGENFUNCTIONS SUPPORT 3D, VOLUMETRIC MATERIAL CLASSIFICATION?

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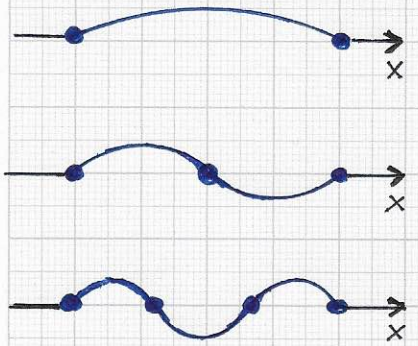
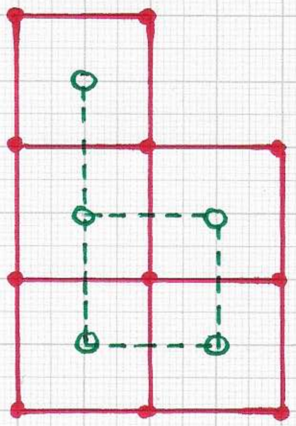
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions:



Sketch of eigenmodes / eigenfunctions of string; in the simple univariate setting on a line, the eigenfunctions $f(x)$ satisfy $-f''(x) = \lambda f(x)$.



2D setting: connectivity graphs implied by given pixels (solid lines) and constructed via center-to-center dual edges (broken lines)

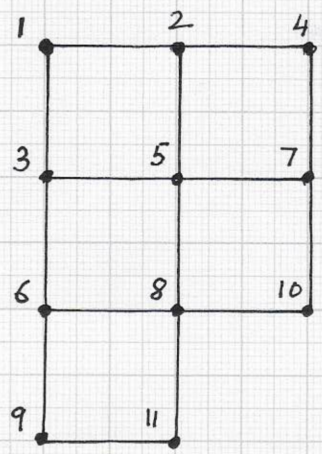
Considering the simple example of a violin string, the string's "eigenmodes" or "eigenfrequencies" are the solution of a second-order differential equation that leads to an eigenvalue problem: one must find eigenfunctions that are "functions that are multiples of their second derivatives", i.e., functions f that satisfy the condition $-\Delta f = \lambda f$. In the simple case of a string, the resulting eigenmodes are trigonometric sin (cos) functions.

How can one "transfer" such an eigenfunction approach to volumetric material scan data characterization?

- Our object/material in 3D space is a set of voxels.
- Our representation is a discrete representation defined by the voxel neighborhoods and voxel-specific densities.
- A connectivity graph of the voxel data set can be established by using voxel corners as vertices and voxel edges as graph edges or by using voxel center points as vertices and edges connecting neighbor voxel center points as graph edges.

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions:



Possible optimal indexing of 11 vertices of a 2D grid; vertices on the boundary require special case treatment when certain partial derivatives must be approximated via a finite difference formula, for example.

(Ref.: Cuthill, McKee, Reducing the bandwidth of sparse symmetric matrices)

– Material properties (densities) can be associated with grid vertices or grid cells, depending on the requirements of the computational technique used to solve the eigenvalue problem. (The connectivity graph defines the grid.)

– We must determine a "very good" indexing/numbering of the grid elements (vertices, cells), to support efficiency for eigenvalue computations.

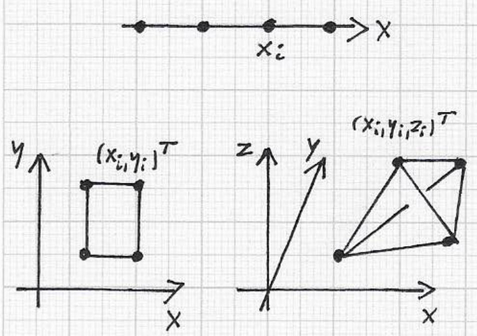
– (It is assumed that) a small number of eigenvalues computed for an object/a set of voxels/a segment can be used to define material-specific feature data.

– Computations should be done in a way that ensures that the resulting eigenvalue features are independent of an object's/segment's geometry and volume; generally, the eigenvalue features should characterize the material and not the geometry of an object.

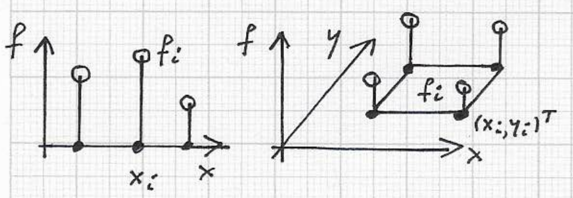
– In some applications, one is interested in the geometry and topology (critical point behavior), e.g., for shape classification and retrieval.

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

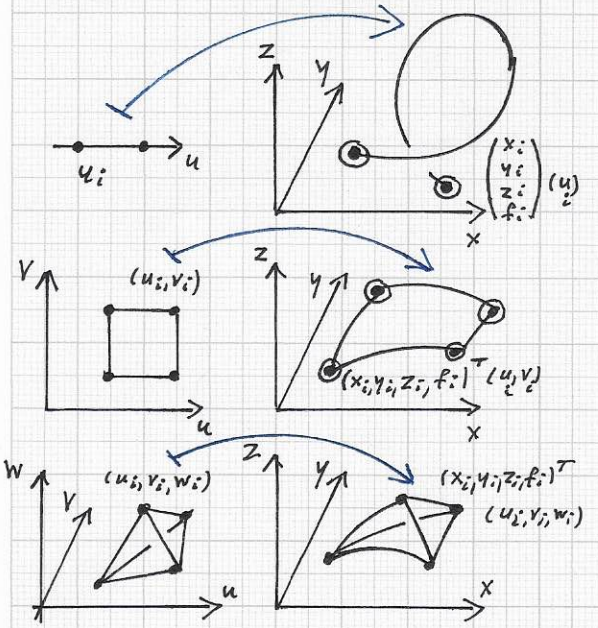
• Laplacian eigenfunctions:



Geometry only: points on line, in plane, in space – defining a finite 1D, 2D, 3D manifold.



Geometry and one dependent variable: points on line, in plane, in space with associated value f.



Parametric definition: points $(x_i, y_i, z_i)^T$ with dependent value f_i in 3D space, mapped from parameter tuples (u_i), (u_i, v_i), (u_i, v_i, w_i).

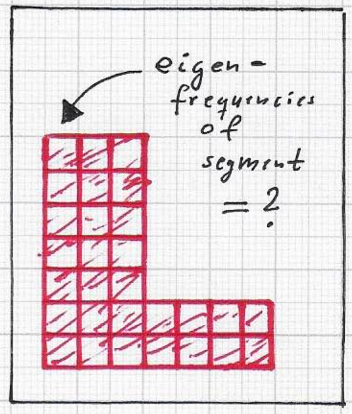
Note: COMSOL is a multiphysics simulation software package commonly used for structural and solid mechanics problems. It is suitable for computing the "eigenmodes"/"eigenfrequencies" of a solid defined by a set of cells (cubes, tetrahedra etc.) with specific material properties (e.g., density, mass etc.).

The Laplacian and Laplace-Beltrami operator can be used to set up a physics-based model of the eigenfrequencies of a complex geometrical shape – e.g., a 3D, volumetric solid with non-homogeneous material density (= a voxel representation of a 3D segment with varying density).

The computation of the eigenfrequencies of a 1D, 2D or 3D manifold in 3D space (an idealized needle, plate or deformable brick) requires one to carefully define the representation of the data: points/vertices, grids, indexing, ...

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions:



Depiction of object/object-segment (≅ set of connected voxels in 3D scan). Cells "belong" to a specific segment when they satisfy a segmentation criterion, e.g., a threshold for a single scalar quantity like density or a more complicated condition based on several quantities.

Ref.: Dongmei NIU et al., Two-dimensional shape retrieval using the distribution of extrema of Laplacian eigenfunctions.

We review the notation used to represent the data needed for the computation of Laplacian eigenfunctions for 1D, 2D and 3D manifolds. (See illustrations on previous page.) (The Laplacian operator has also been used as "graph Laplacian" in a more "abstract way"; nevertheless, if it was possible to incorporate a quantity like "density" into the graph Laplacian, this method could be helpful for classification.) In our specific application – feature value computation for the voxels defining a "segment" in the domain of a 3D scan – we would like to apply the Laplace/Laplace-Beltrami operator to finite volumes represented by a set of voxels (and/or other simple types of volumetric cells), with an associated material property per voxel. The operator and the computational modeling set-up must be defined and used such that an eigenvalue problem produces the "eigenfrequencies" (eigenvalues and eigenvectors) of a segment. It can be assumed that a small number of segment-specific eigenvalues defines a material-class signature, useful for classification.

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