Stratovan

■ OBJECT AND MATERIAL EIGENFUNCTION FEATURES – Cont'd.

- **Laplacian eigenfunctions:**
- **Newton:**
  $\vec{F} = m\,\vec{a} = m\ddot{x}$
- **Hooke:**
  $\vec{F} = -kx$

$\Rightarrow$ $m\ddot{x} = -kx$

$\Rightarrow$ 1D case, for example: "A trigonometric function's second derivative is a multiple of the function itself."

- **Laplace operator:**
  $\Delta f = \nabla^2 f = \nabla \cdot \nabla f$
  $= \operatorname{div} \operatorname{grad} f$
  - 1D: $\Delta f = f''(x)$
  - 2D: $\Delta f = \frac{\partial^2}{\partial x^2} f(x,y) + \frac{\partial^2}{\partial y^2} f(x,y) = f_{xx} + f_{yy}$
  - 3D: $\Delta f = f_{xx} + f_{yy} + f_{zz}$

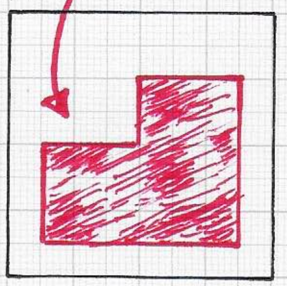
What is the potential value of computing and using eigenfunctions/eigenvalues of the Laplacian of a 3D, volumetric, solid object of possibly varying density? In our case why might it be helpful to compute Laplacian characteristics of a segment, a set of voxels constituting a compact, connected object? In the field of solid and structural mechanics the understanding of an object's "harmonic behavior" is of great importance. These are just some of the phenomena, terms and equations/laws that are relevant:

Newton's law, Hooke's law, wave equation, Laplace/Poisson/Helmholtz equation, resonance, vibration, harmonics, longitudinal/transversal wave, eigenfrequency/eigenmode, spectral analysis, eigenvalue/eigenvector/eigenfunction, quasi-harmonic partial differential equation (for inhomogeneous, anisotropic media/objects), mass-spring model, mass matrix, stiffness matrix, finite difference/element method, ...

OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions:

○ 1 segment



Object = 3D segment = set of voxels with different density (or mass) values, defining inhomogeneous, anisotropic solid. Compute Laplacian eigenvalues for this solid. Eigenvalues could characterize material class and also shape type (compact ball-like object vs. thin sheet-like object).

In our application – object/material classification – we are NOT interested what the true physically "correct eigenfrequencies" of a 3D segment, i.e., a set of voxels, are; we are only interested in "good features for classification" that characterize the 3D segment's material properties (and, to a much lesser degree, its geometry/shape) with a "unique signature" – if possible. The Laplacian eigenvalues of a 3D segment, when treated like a 3D object with material-specific eigenfrequency behavior, could potentially define very useful features.

The eigenfunctions $f(x,y)$, not depending on time, of a homogeneous 2-manifold membrane (ideal) are defined by the eigenvalue problem (Helmholtz equation)

$$-\Delta f(x,y) = \lambda f(x,y)$$

$$\Leftrightarrow -(f_{xx} + f_{yy}) = \lambda f(x,y) .$$

○ **Knots, knot lines, knot surfaces:**

The Laplacian eigenfunctions of a 1-manifold curve, of a 2-manifold surface, of a 3-manifold solid have KNOTS that are point knots, knot curves, knot surfaces, respectively.

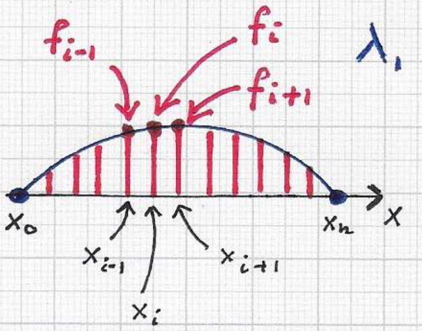
The corresponding trivariate eigenfunctions of a homogeneous 3-manifold solid satisfy

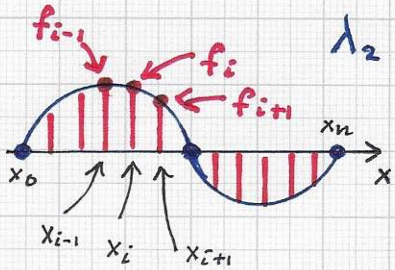
$$-(f_{xx} + f_{yy} + f_{zz}) = \lambda f(x,y,z) .$$

○ **Cladni figures** visualize the "nodal lines" of a membrane's eigenfunctions.

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• **Laplacian eigenfunctions:** We briefly review basic principles used in finite difference methods for numerically solving (partial) differential equations. The goal of these methods is the computation of a discrete approximation of an unknown function f that must satisfy defined differential properties. The discrete approximation of the unknown function is eventually represented by a set of points/vertices in the function's domain (usually connected via a grid) and computed function values at these points. The governing (partial) differential equation(s) that must be satisfied are "reduced" to locally expressed finite difference formulas that approximate the respective differential property of the function f at a (grid) point. Thus, the number of unknown values that must be computed is (defined by) the number of points in f's domain – with one function value to be computed for each grid point. In the case of the Helmholtz equation, defining an eigenvalue problem, many discrete eigen function approximations result.



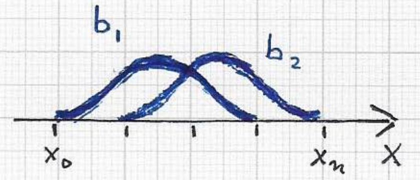


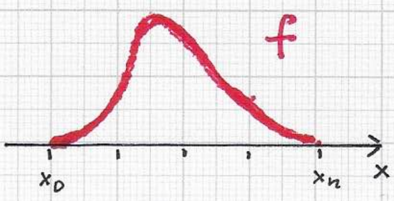
1D, univariate case of discrete eigen function computation based on Helmholtz equation. The 1D domain is discretized via points x_i; finite difference formulas are employed to locally approximate a function's derivatives in terms of (unknown) values f_{i-1}, f_i, f_{i+1} etc. (The used underlying local polynomial approximation must be at least quadratic.)

The Helmholtz equation, together with the used domain discretization and finite difference method, defines eigen values $\lambda_1, \lambda_2, \ldots$, with associated eigenfunctions represented via discrete "eigen-vectors" $(f_0, f_1, \ldots, f_n)^T$.

■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions:





Finite element method producing continuous function f as a linear combination of two smooth, continuous basis functions b_1, b_2:

$$f(x) = \sum_{i=1}^{2} c_i \, b_i(x) .$$

Finite difference methods only generate discrete function values at a finite number of grid vertices.

• Note: Finite difference methods (in their native form) compute a discrete, point-wise approximation of an unknown function(s) for which specified differential equations must hold. Finite element methods, in contrast, produce a continuous approximation; the unknown function is expressed as a linear combination of continuous basis functions b_i and "optimal" coefficients c_i are computed to obtain the unknown function(s) as $f = \sum_i c_i \cdot b_i$. For example, when dealing with simple Cartesian-type grids, the basis functions b_i can be chosen as tensor product B-spline basis functions of an order that suffices to represent all derivatives in the given (partial) differential equations.

BUT: Our driving application is the computation of features of 3D objects/segments. Thus, we are merely interested in the potential value of the eigenvalues of the Helmholtz equation as material/object features. The associated eigenfunctions are only of secondary interest.

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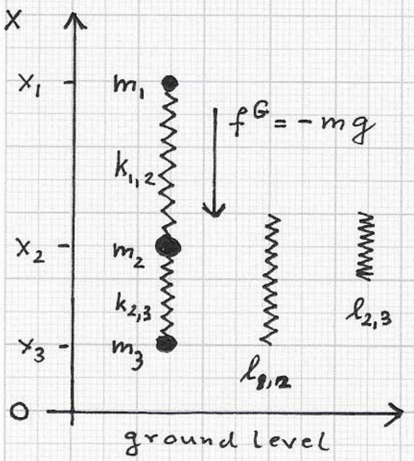
■ OBJECT AND MATERIAL EIGENFUNCTIONS – Cont'd.

• Laplacian eigenfunctions:



Simple 3-point mass-spring model for points with masses m_1, m_2, m_3. Points 1 and 2 have masses m_1, m_2 and are connected by a spring with spring constant $k_{1,2}$ and non-deformed state length $l_{1,2}$. Points 2 and 3 are connected by a spring with spring constant $k_{2,3}$ of length $l_{2,3}$ when not deformed. The gravitational force f^G is given as $f^G = -mg$.

3 gravit. forces:

$$f_i^G = -m_i g, \quad i=1\ldots 3$$

2 spring forces:

$$f_{i,i+1}^S = -k_{i,i+1} \cdot (x_i - x_{i+1} - l_{i,i+1}), \quad i=1\ldots 2$$

So-called mass-spring systems are used in physics and physics-based computer animation to simulate the behavior of deformable objects. Continuous objects are discretely represented by points with associated masses and (elastic) springs connecting certain points with each other. Thus, such a mass-spring model can be established for the kind of 3D object/segment representations we use: a set of voxels with associated masses and a set of edges (="springs") between pairs of voxels sharing a face. For our purposes we are interested in the eigenmodes/eigenfrequencies of such a model to use the eigenvalues as potential segment features.

Prior to discussing the eigenfrequency problem for such a mass-spring model it should be helpful to review a simple example for three points (with masses) connected in a "linear chain" by springs (with spring constants), exposed to constant gravitational force (see figure). The laws of Lagrangian motion & linearly deformable elastic springs describe this scenario.